Static Program Analysis

Data Flow Analysis — Foundations

Nanjing University

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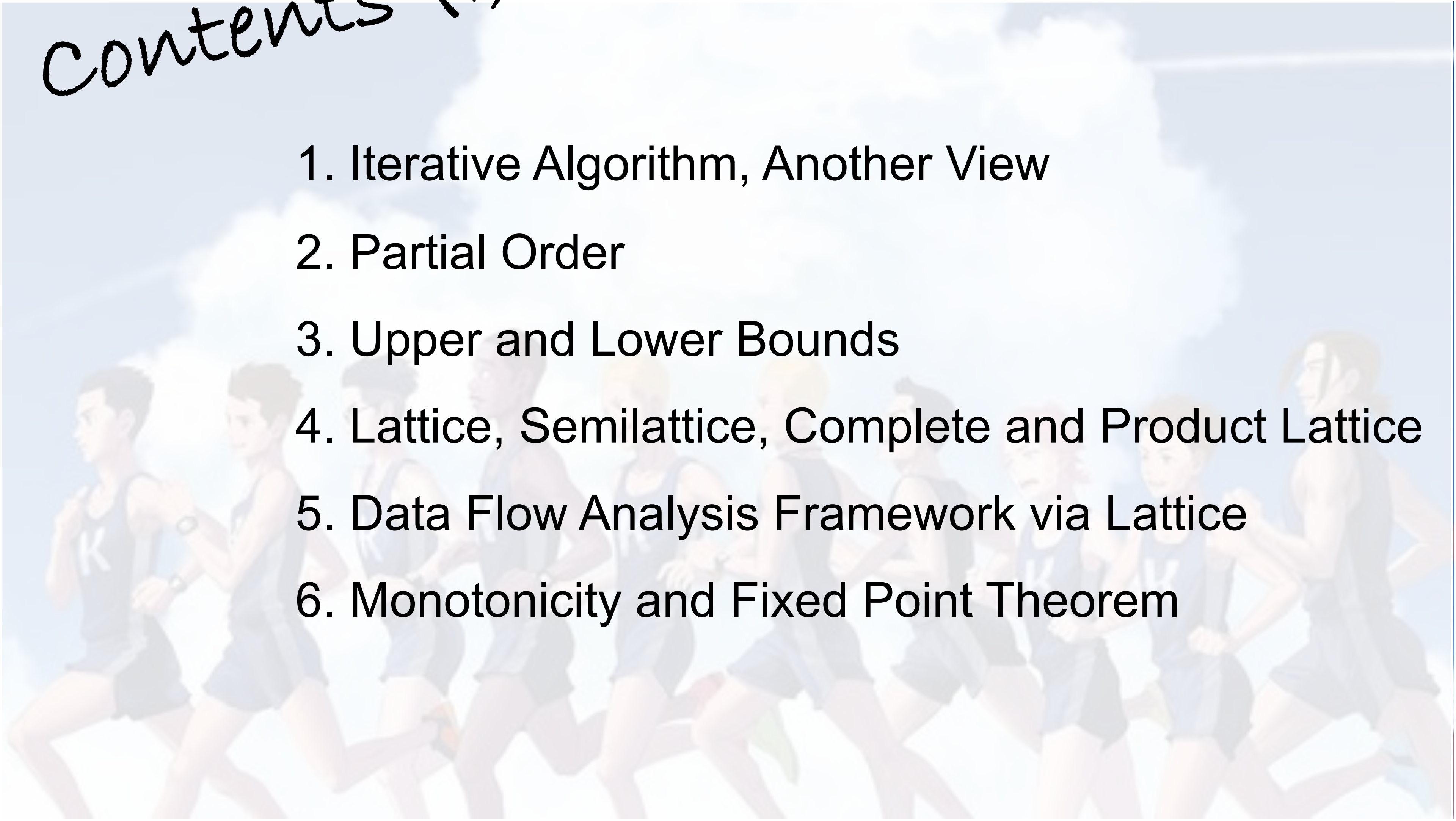
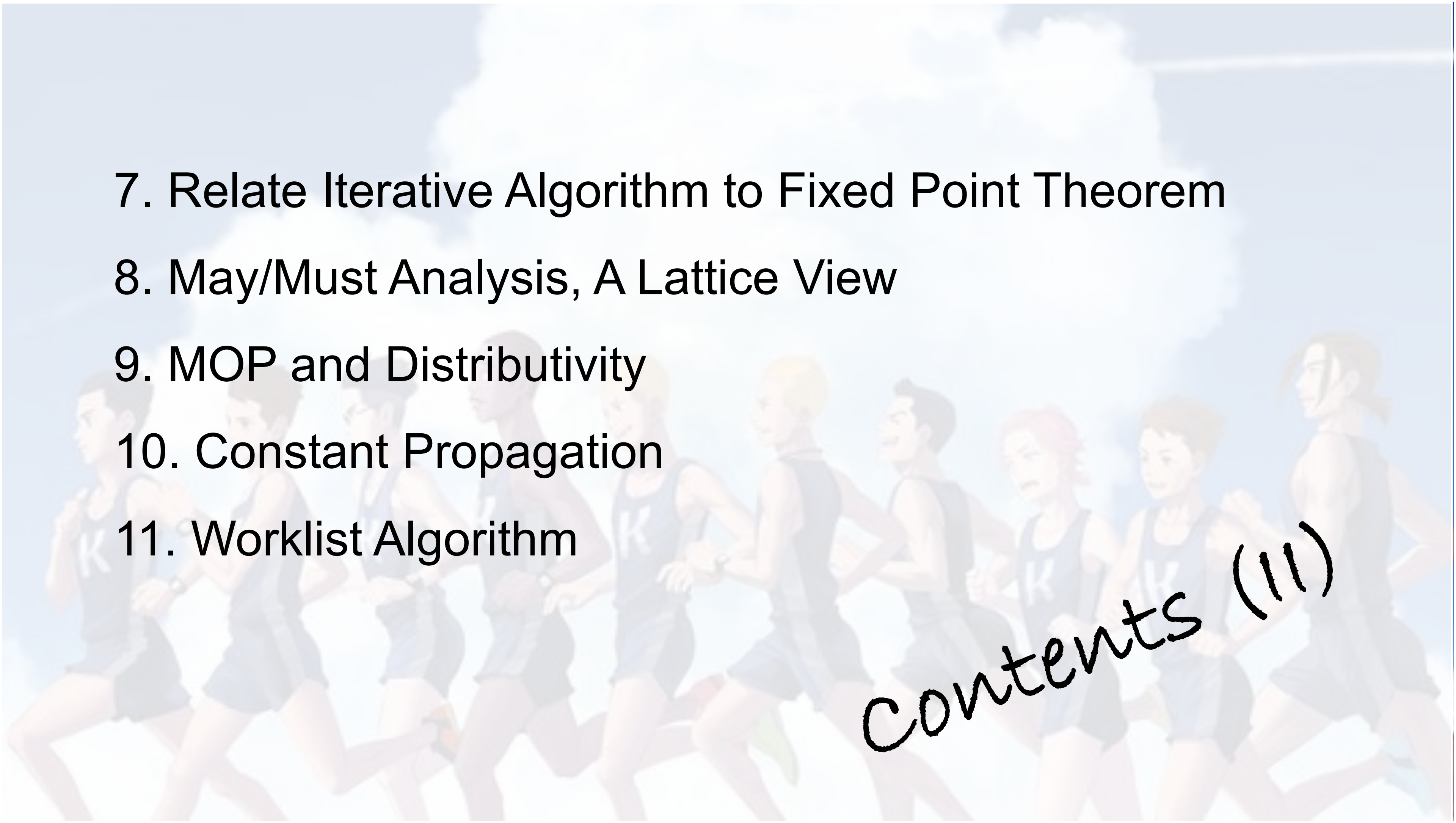
2020

Contents (1)



1. Iterative Algorithm, Another View
2. Partial Order
3. Upper and Lower Bounds
4. Lattice, Semilattice, Complete and Product Lattice
5. Data Flow Analysis Framework via Lattice
6. Monotonicity and Fixed Point Theorem



7. Relate Iterative Algorithm to Fixed Point Theorem
8. May/Must Analysis, A Lattice View
9. MOP and Distributivity
10. Constant Propagation
11. Worklist Algorithm

Contents (II)

Let us first recall the iterative algorithm for data flow analysis

This general iterative algorithm produces a solution to data flow analysis

Iterative Algorithm for May & Forward Analysis

INPUT: CFG (*kill*$_B$ and *gen*$_B$ computed for each basic block *B*)

OUTPUT: IN[*B*] and OUT[*B*] for each basic block *B*

METHOD:

```
OUT[$entry$] = $\emptyset$;
for (each basic block $B\backslash entry$)
    OUT[$B$] = $\emptyset$;
while (changes to any OUT occur)
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        IN[$B$] = $\bigcup_{P \text{ a predecessor of } B}$ OUT[$P$];
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View Iterative Algorithm in Another Way

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$$(\mathrm{OUT}[n_1], \mathrm{OUT}[n_2], \ldots, \mathrm{OUT}[n_k])$$

as an element of set $(V_1 \times V_2 \ldots \times V_k)$ denoted as V^k , to hold the values of the analysis after each iteration.

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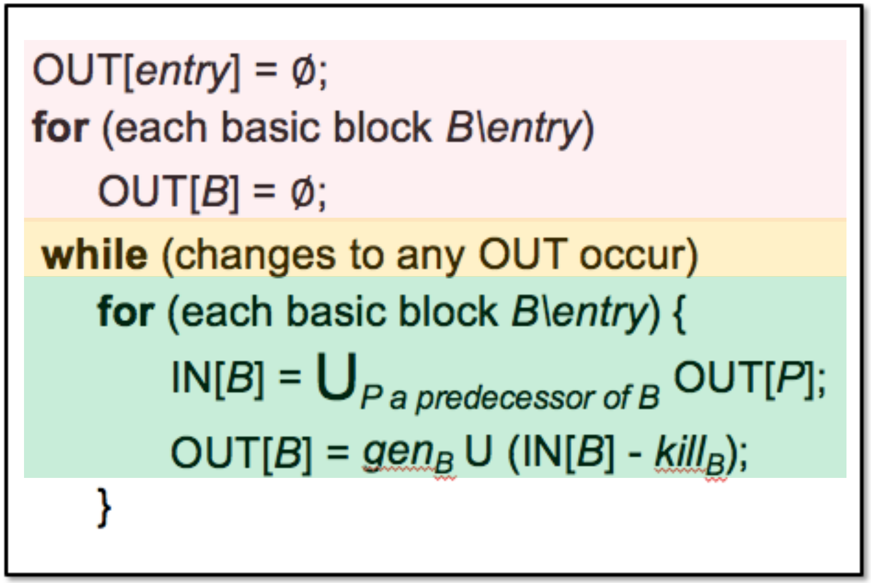
- Then the algorithm outputs a series of k-tuples iteratively until a k-tuple is the same as the last one in two consecutive iterations



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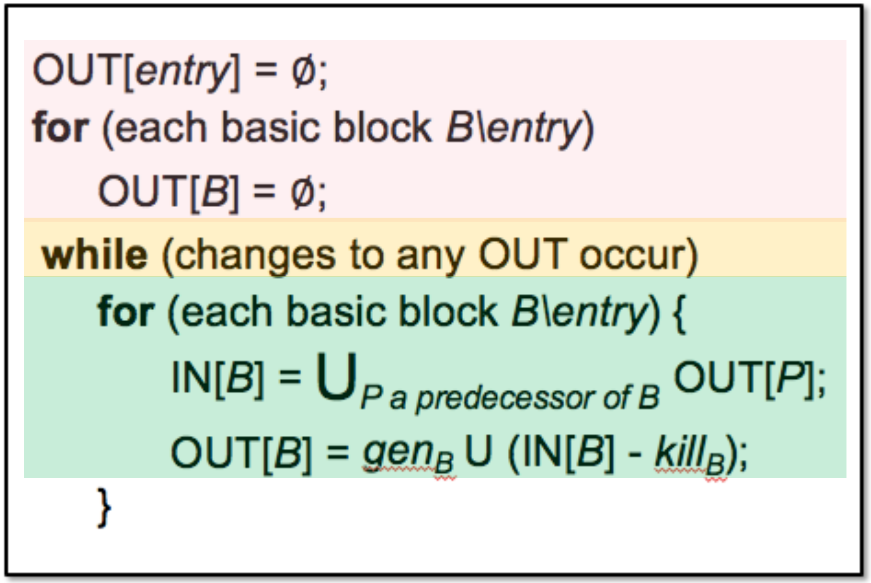

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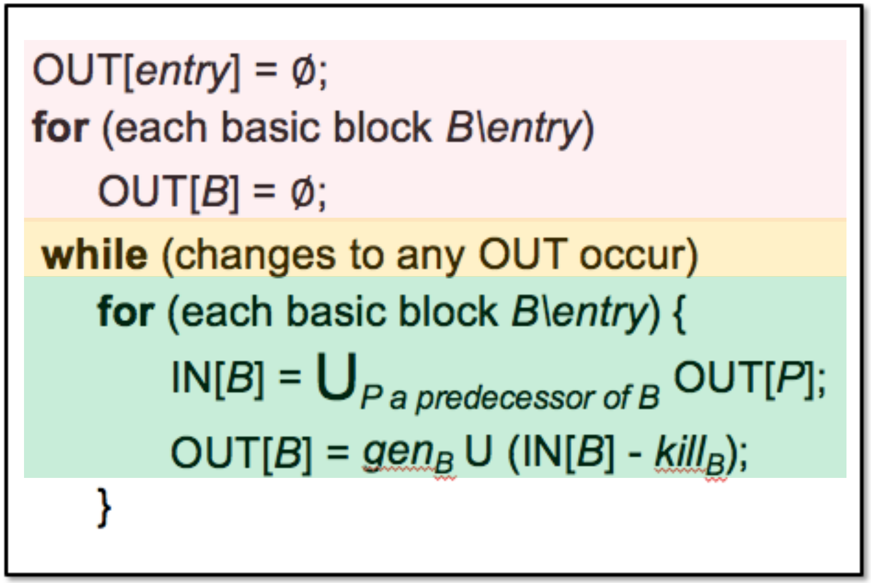

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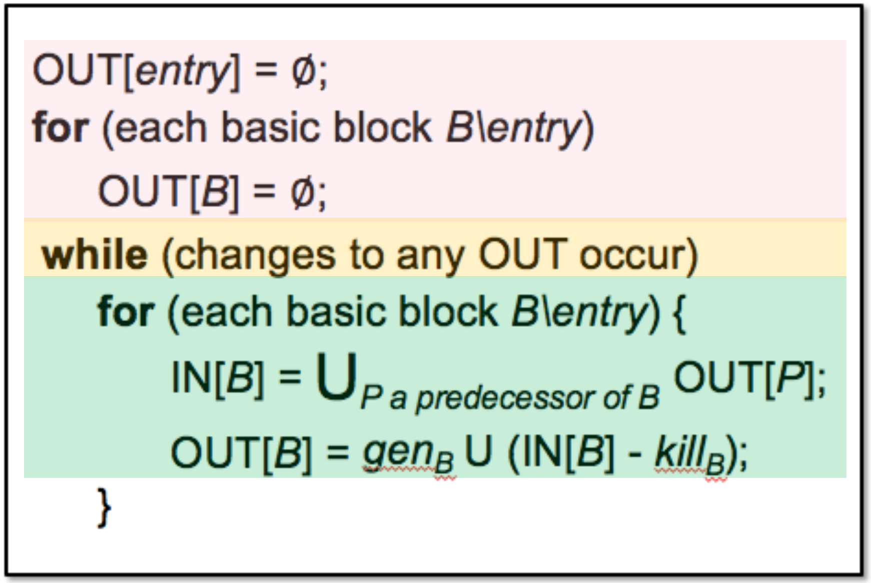

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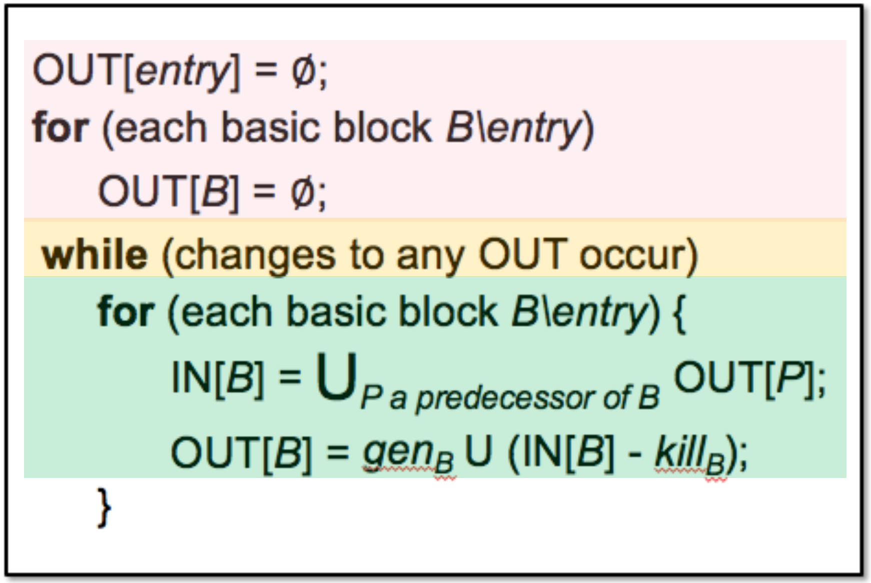

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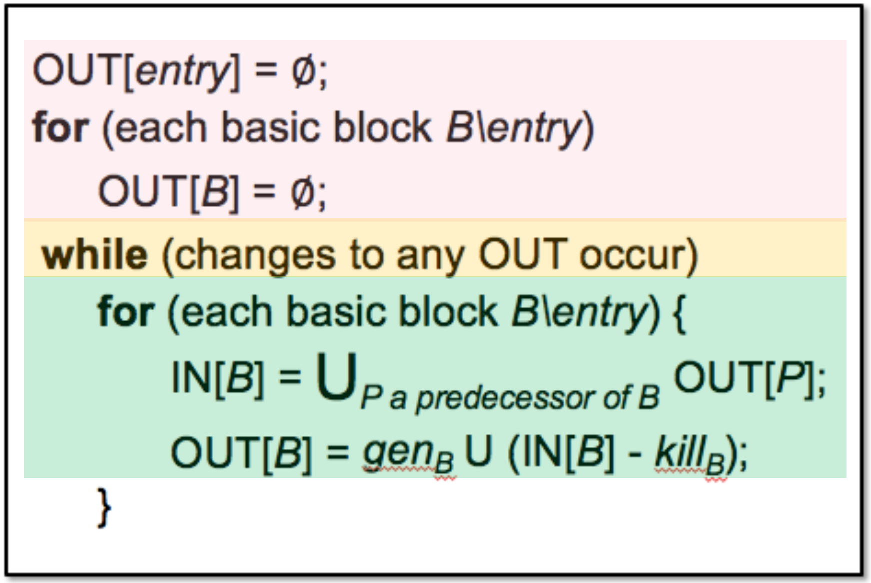

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$$init \rightarrow (\bot, \ \bot, \ \ldots, \ \bot) = X_0$$

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$$\vdots$$

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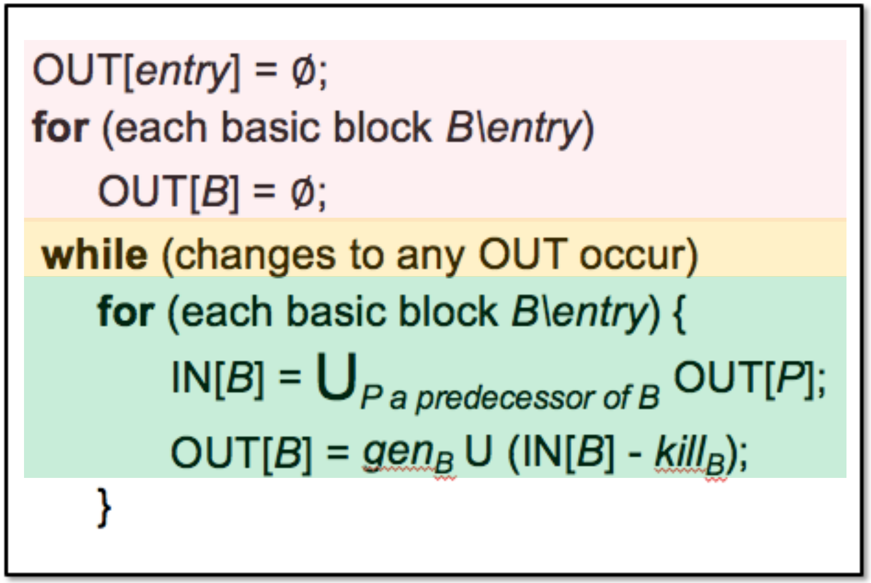

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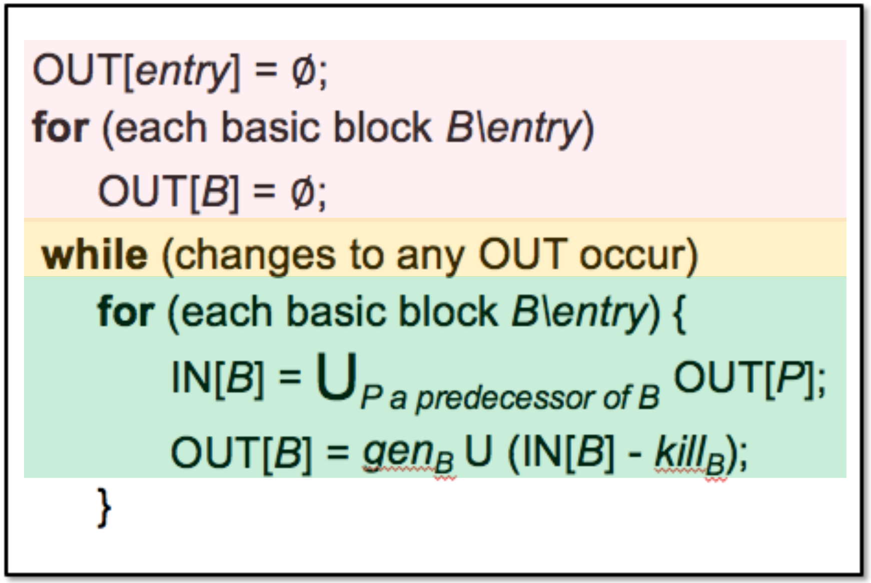

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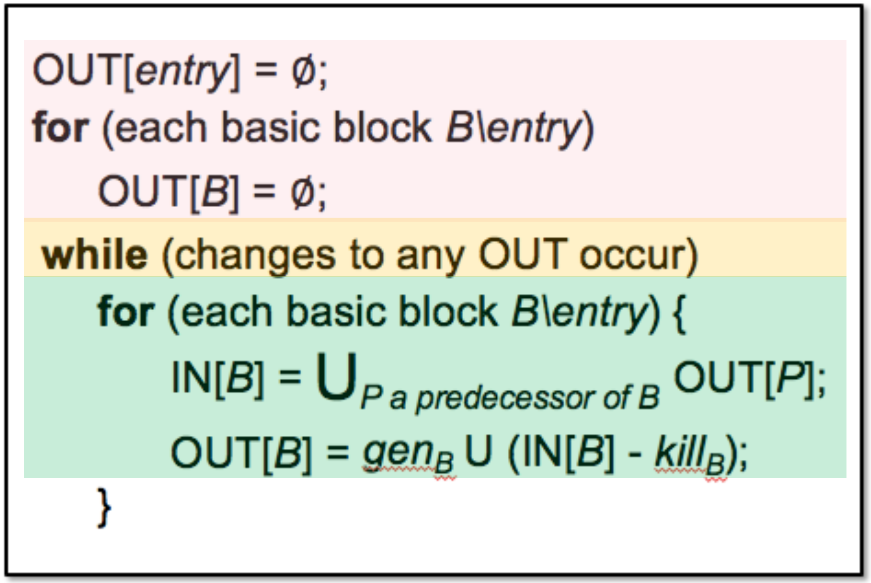

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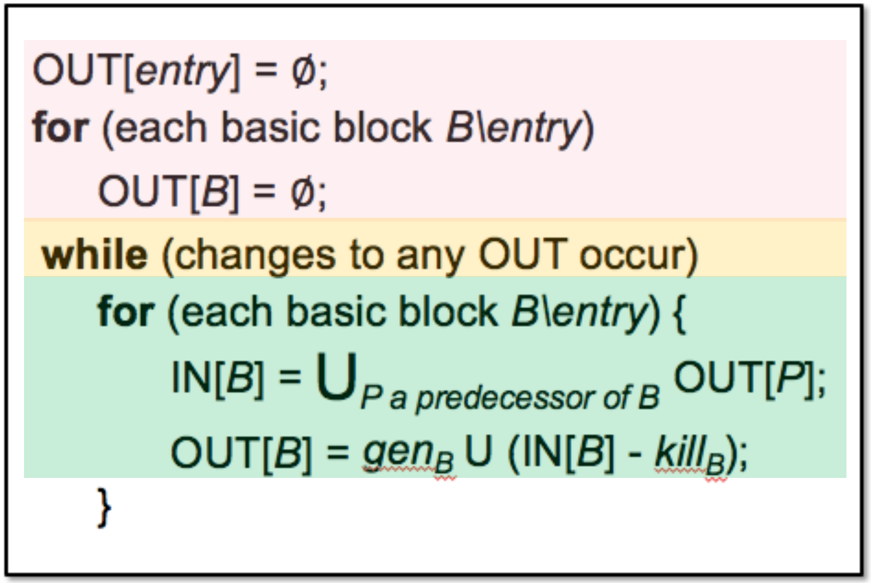

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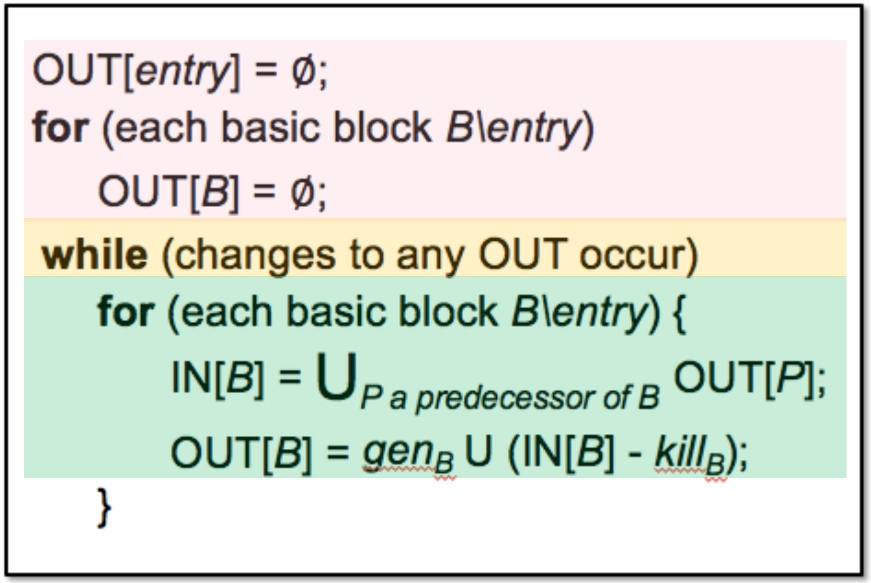

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To answer these questions, let us learn some math first

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(3) *Transitivity* $1 \leq 2 \land 2 \leq 3$ then $1 \leq 3$

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Example 1. Is $(S, \sqsubseteq)$ a poset where S is a set of integers and $\sqsubseteq$ represents $\leq$ (less than or equal to)?

✓ (1) *Reflexivity* $1 \leq 1, 2 \leq 2$

✓ (2) *Antisymmetry* $x \leq y \wedge y \leq x$ then $x = y$

✓ (3) *Transitivity* $1 \leq 2 \wedge 2 \leq 3$ then $1 \leq 3$

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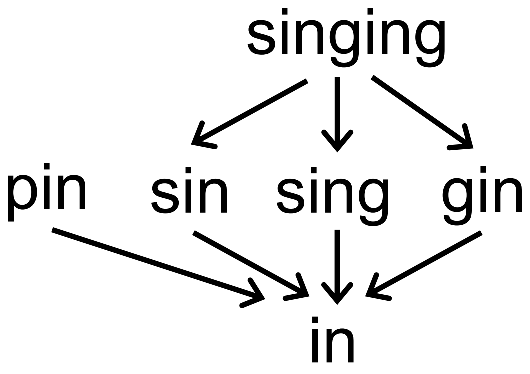

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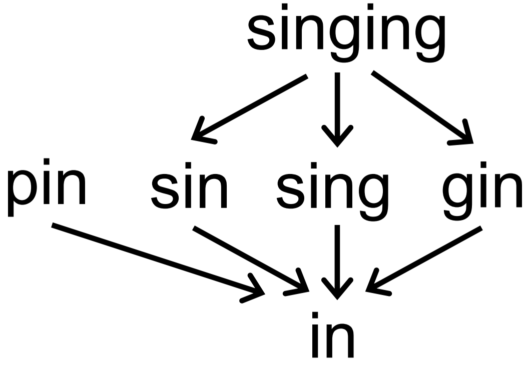

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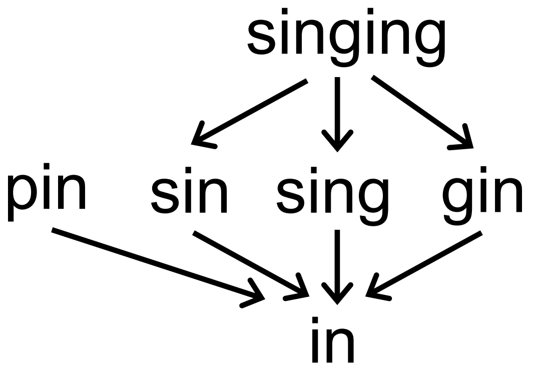

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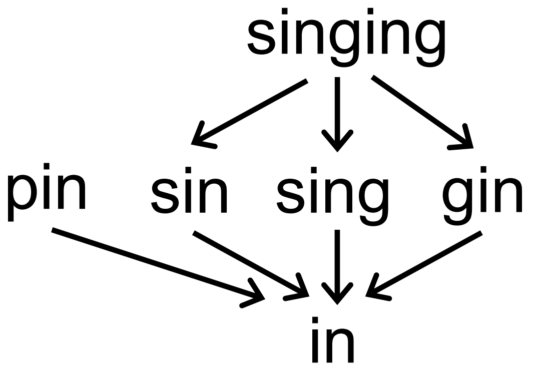

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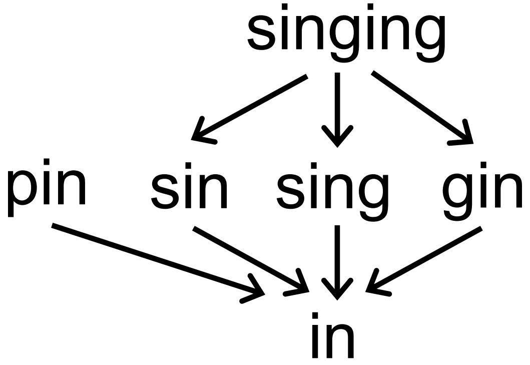

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partial means for a pair of set elements in P, they could be incomparable; in other words, not necessary that every pair of set elements must satisfy the ordering $\sqsubseteq$

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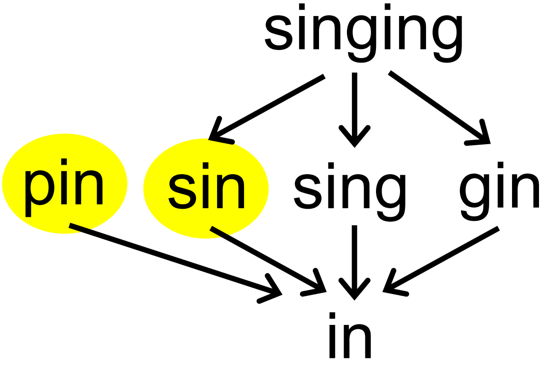

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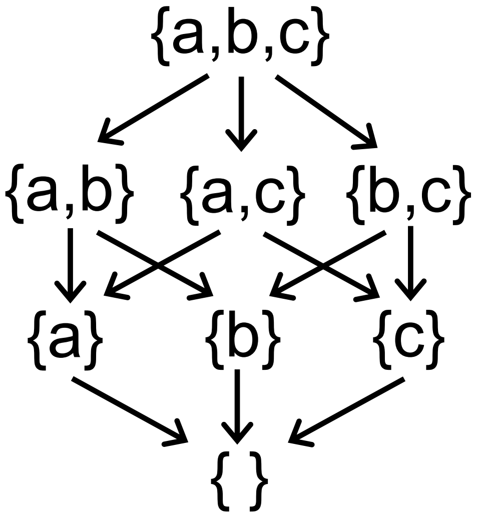

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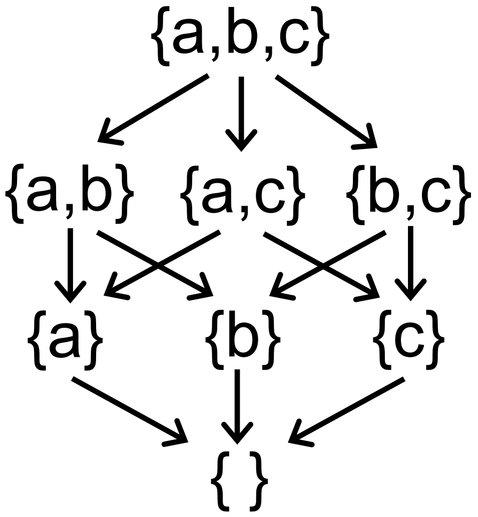

{a,b,c}

{a,b} {a,c} {b,c}

{a} {b} {c}

{ }

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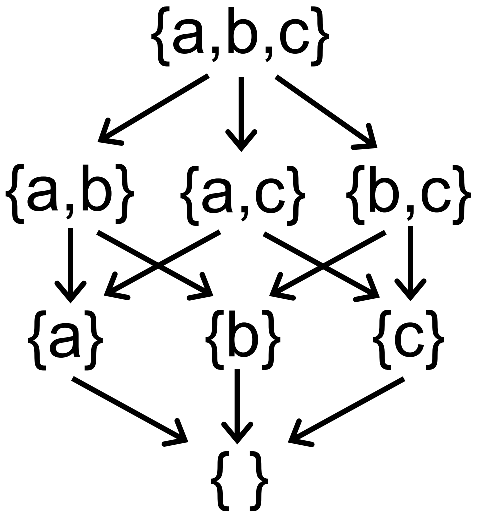

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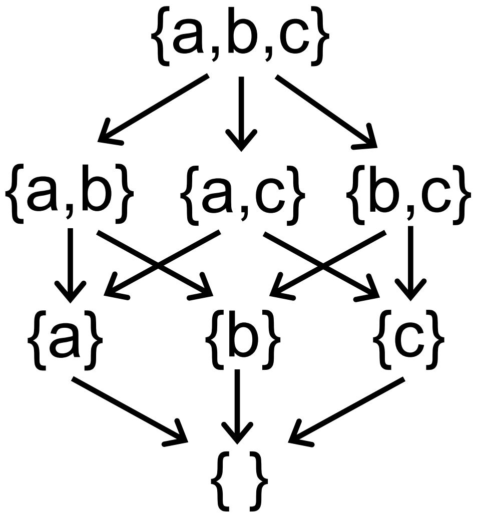

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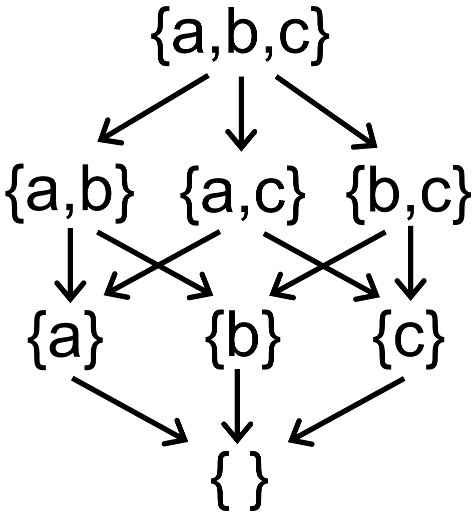

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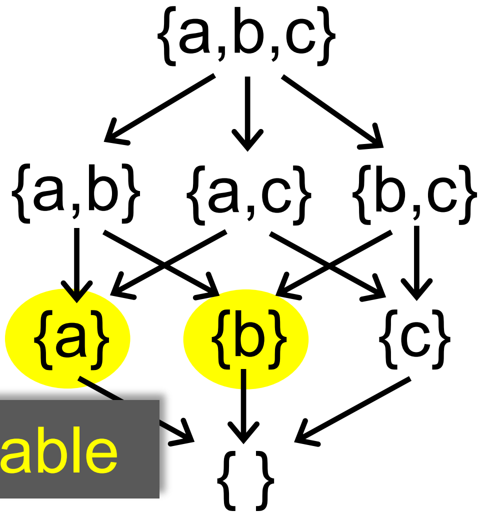

partial → incomparable

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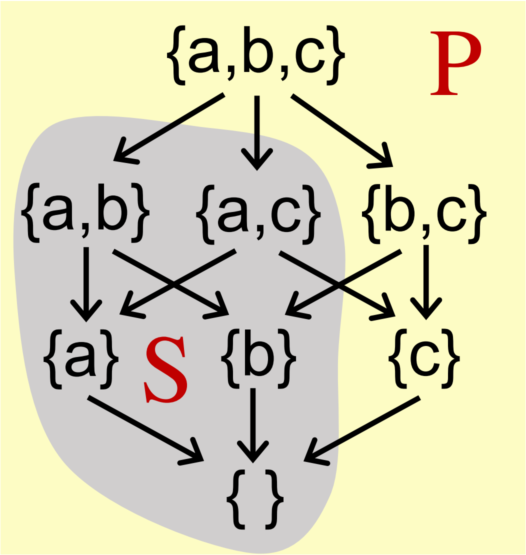

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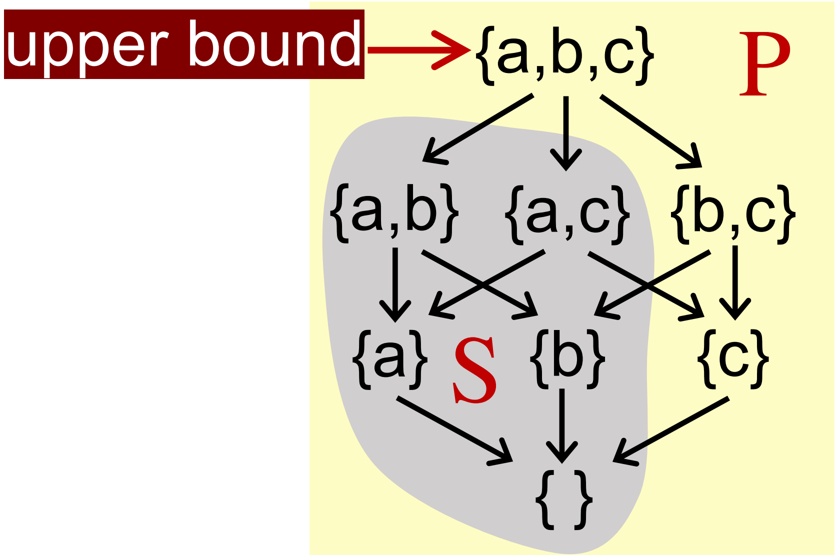

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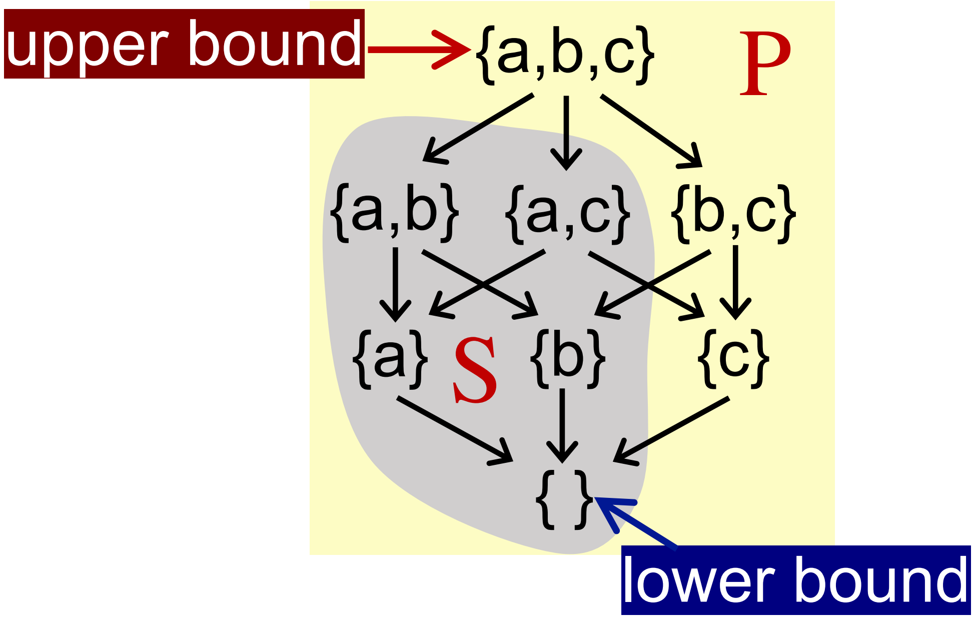

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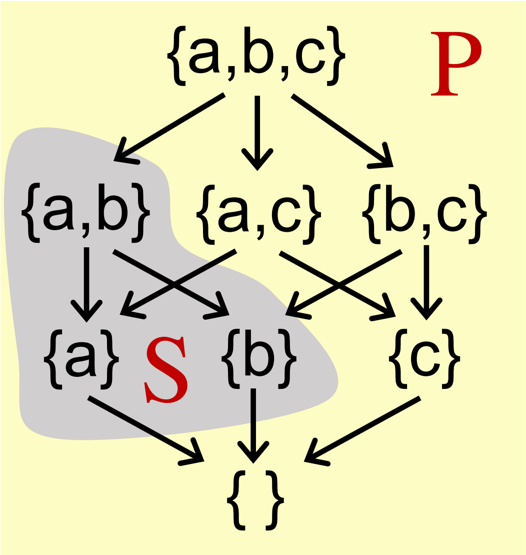

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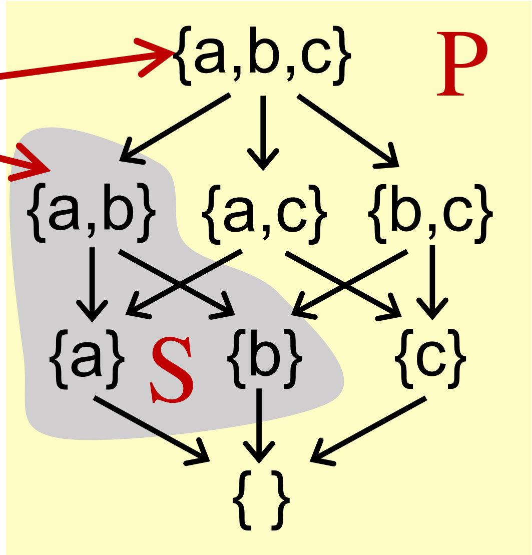
upper bound



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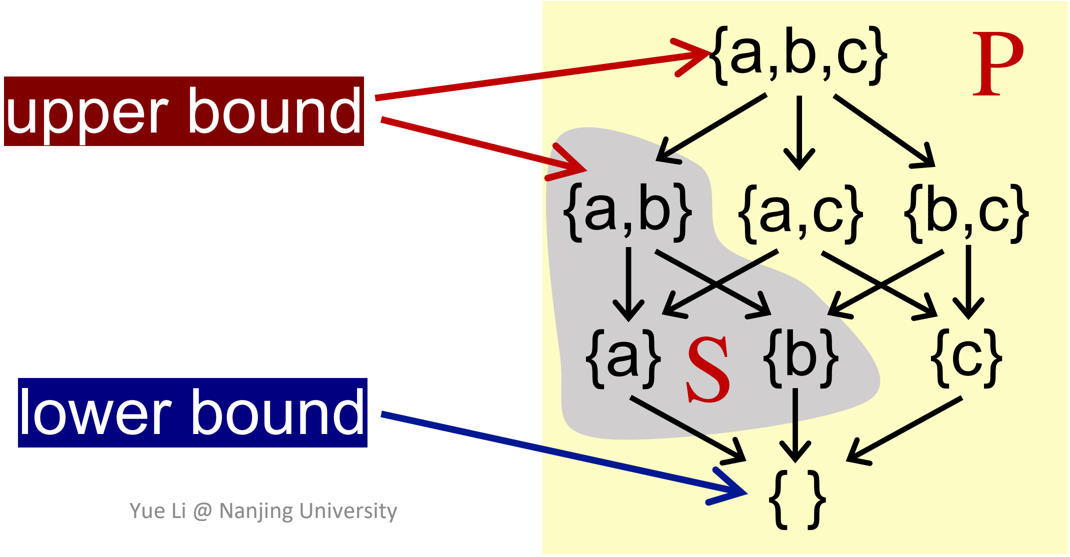

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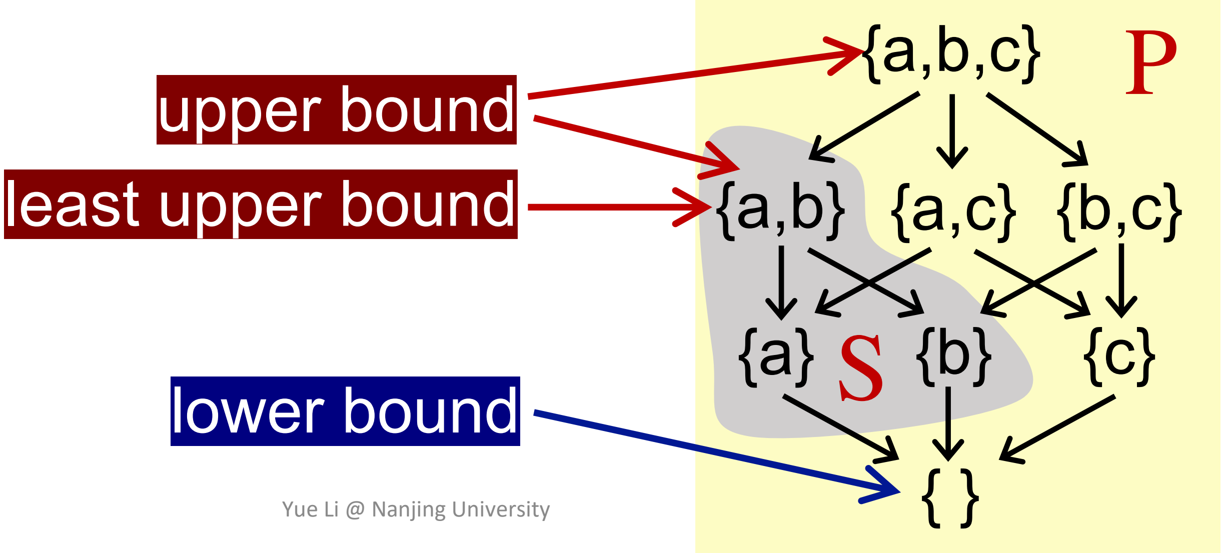

upper bound

least upper bound

lower bound

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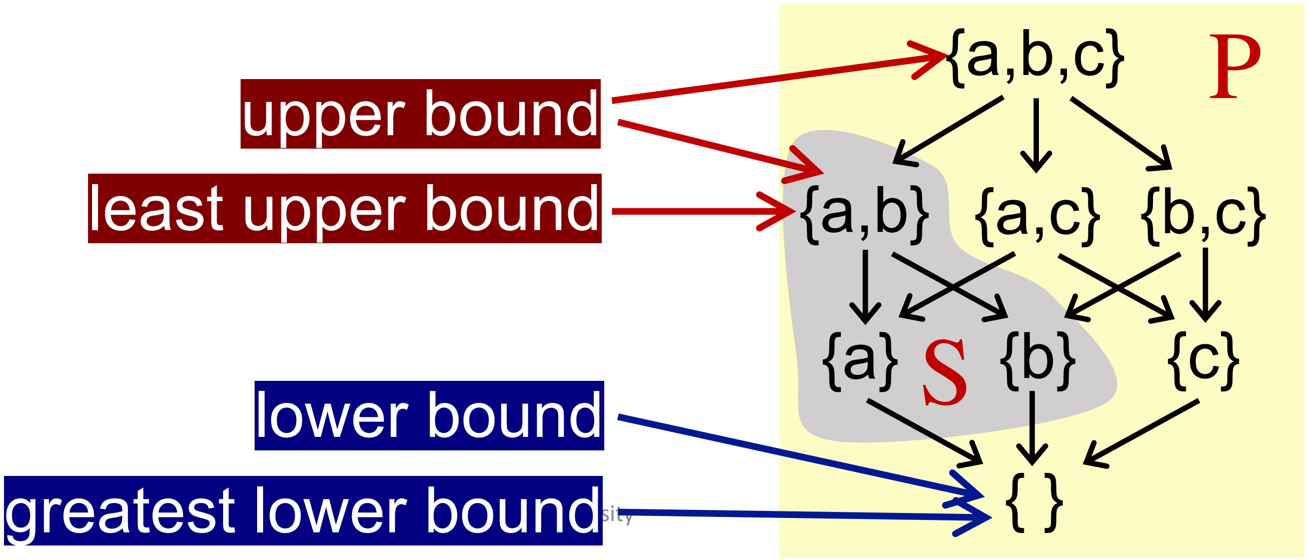

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Usually, if S contains only two elements a and b ($S = \{a, b\}$), then

$\sqcup S$ can be written $a \sqcup b$ (the join of a and b)

$\sqcap S$ can be written $a \sqcap b$ (the meet of a and b)

Some Properties

- Not every poset has *lub* or *glb*

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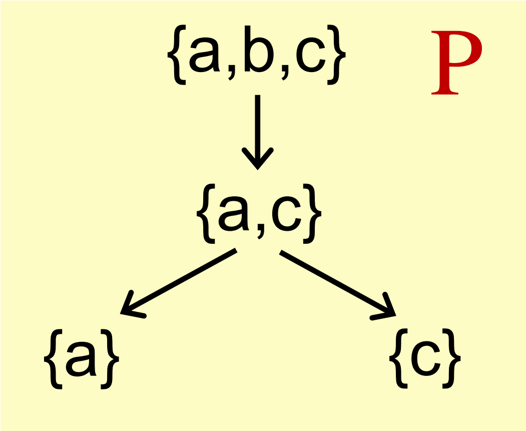

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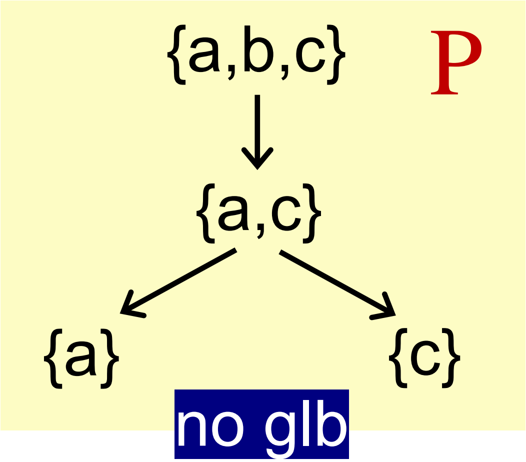

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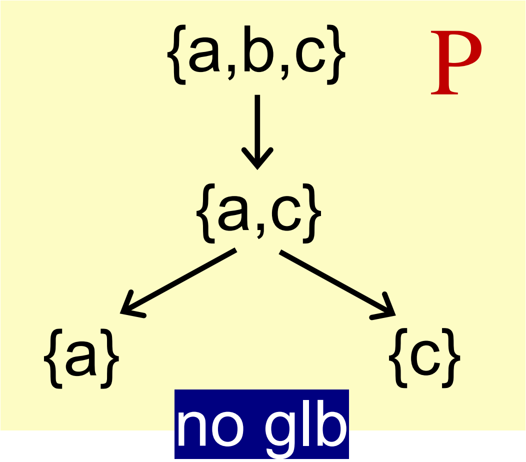

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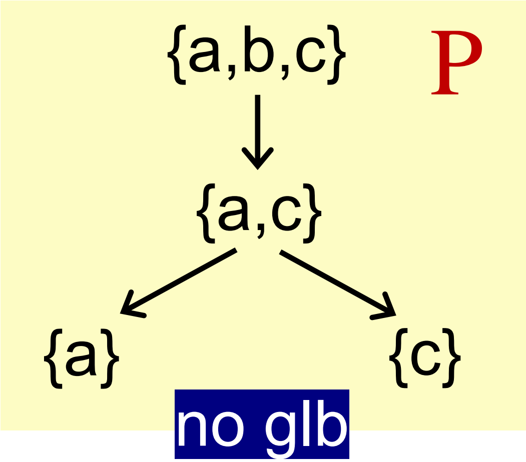

{a,b,c}

P

{a,c}

{a}

{c}

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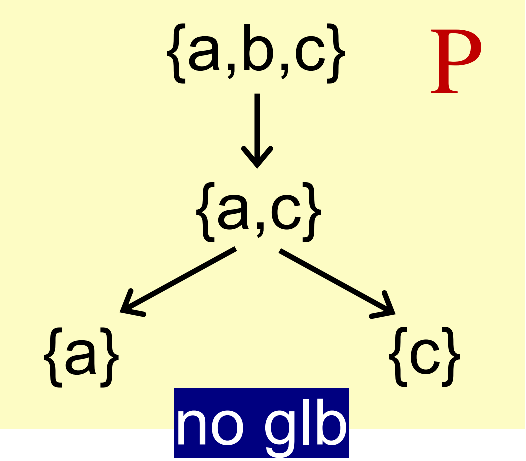

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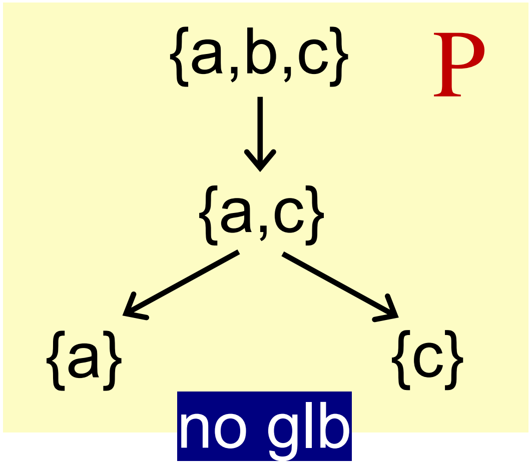

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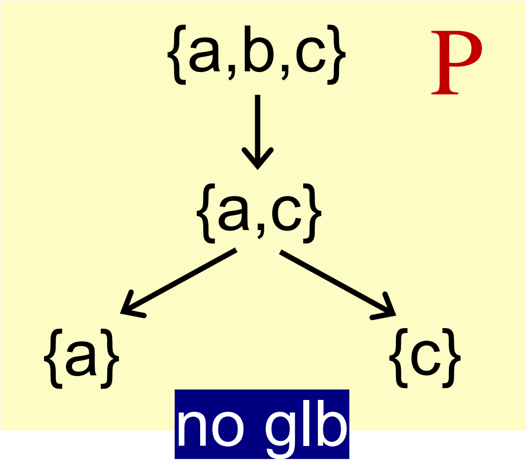

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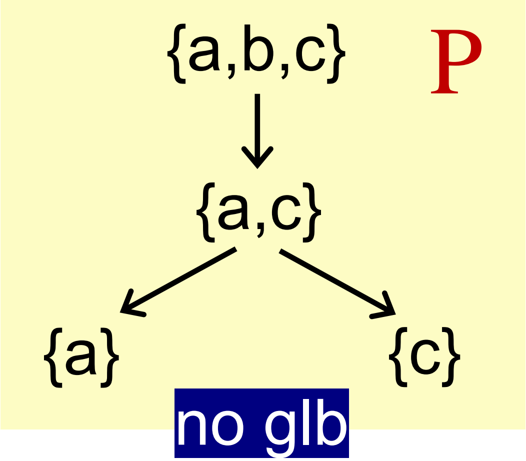

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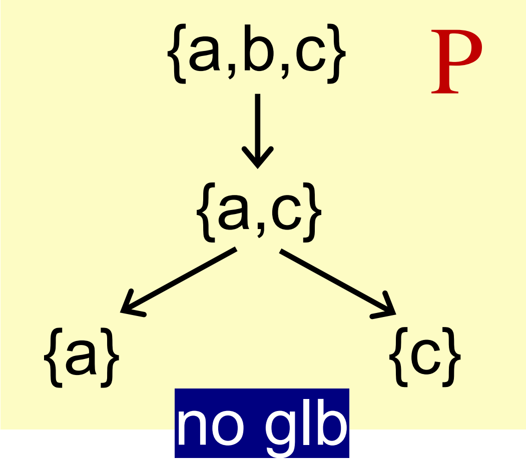

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✓ The $\sqcup$ operator means "max" and $\sqcap$ operator means "min"

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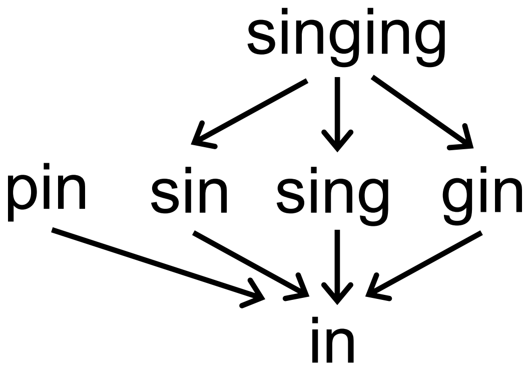

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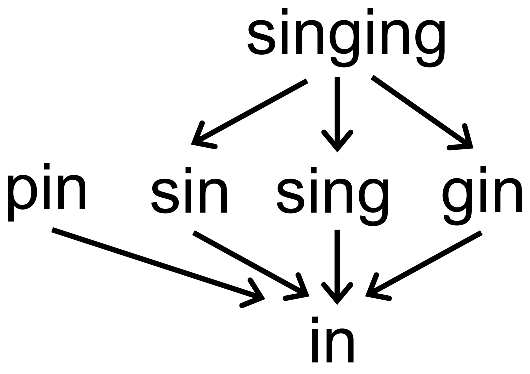

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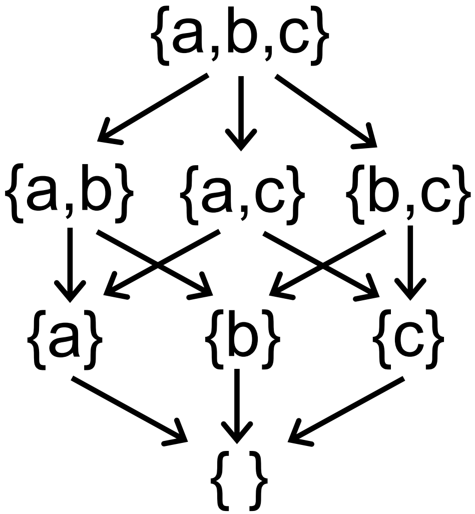

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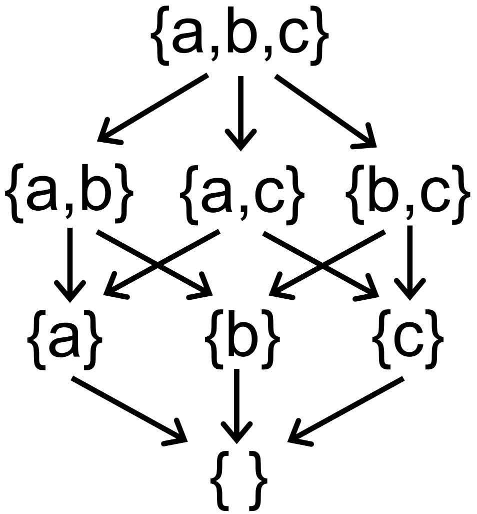

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Semilattice

Given a poset $(P, \sqsubseteq)$, $\forall a, b \in P$,
if only $a \sqcup b$ exists, then $(P, \sqsubseteq)$ is called a join semilattice
if only $a \sqcap b$ exists, then $(P, \sqsubseteq)$ is called a meet semilattice

Complete Lattice

Given a lattice (P, $\sqsubseteq$), for arbitrary subset S of P, if $\sqcup$S and $\sqcap$S exist, then (P, $\sqsubseteq$) is called a complete lattice

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✗ For a subset S^+ including all positive integers, it has no $\sqcup S^+$ ($+\infty$)

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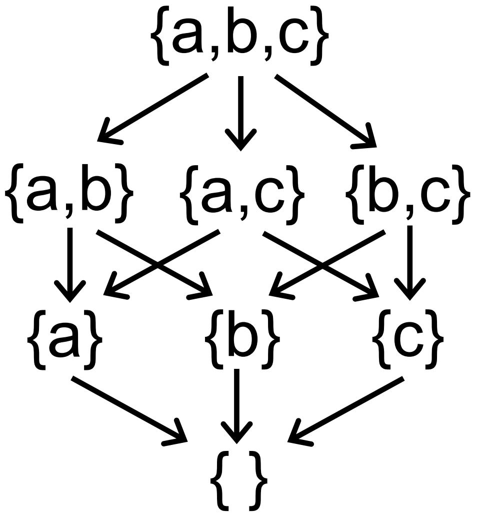

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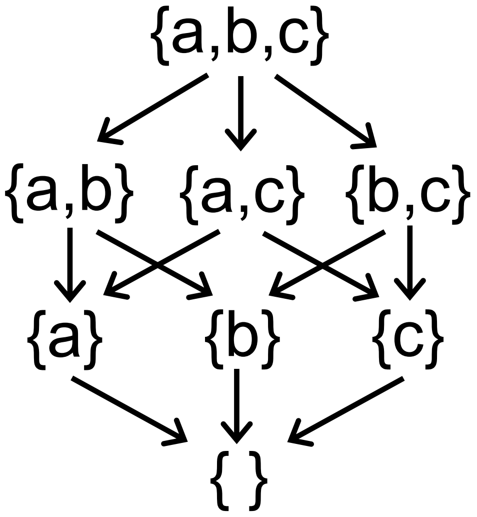
✓ Note: the definition of bounds implies that the bounds are not necessarily in the subsets (but they must be in the lattice)



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a greatest element ⊤ = ⊔P called top and
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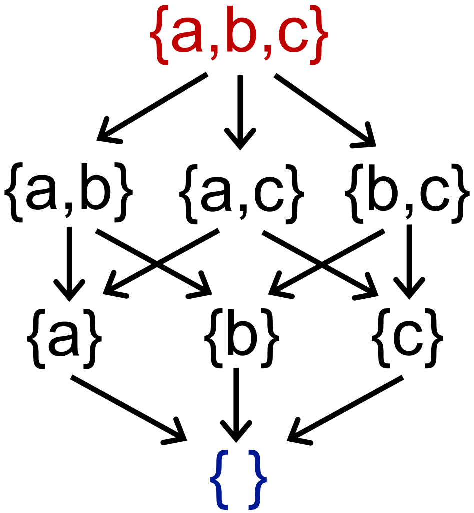

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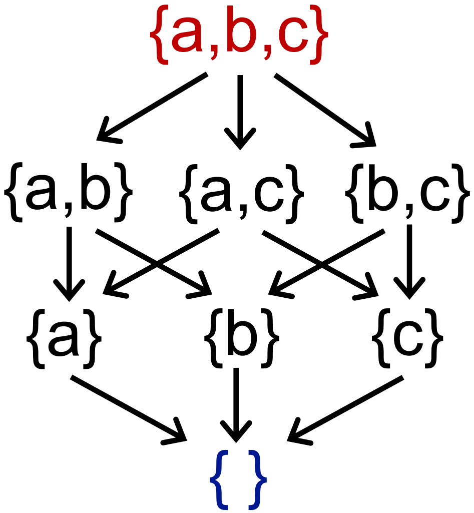

Complete Lattice Mostly focused in data flow analysis

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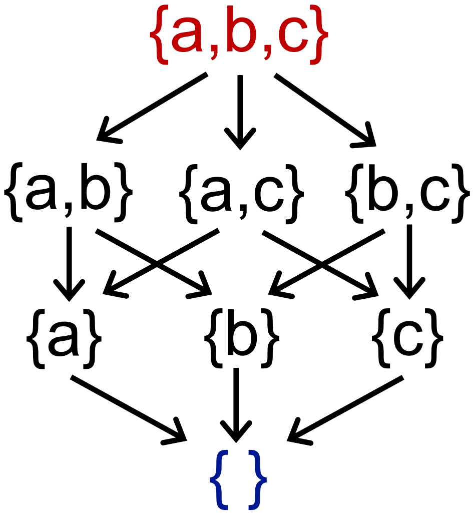

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Given lattices $L_1 = (P_1, \sqsubseteq_1)$, $L_2 = (P_2, \sqsubseteq_2)$, …, $L_n = (P_n, \sqsubseteq_n)$, if for all i, $(P_i, \sqsubseteq_i)$ has $\sqcup_i$ (least upper bound) and $\sqcap_i$ (greatest lower bound), then we can have a product lattice $L^n = (P, \sqsubseteq)$ that is defined by:

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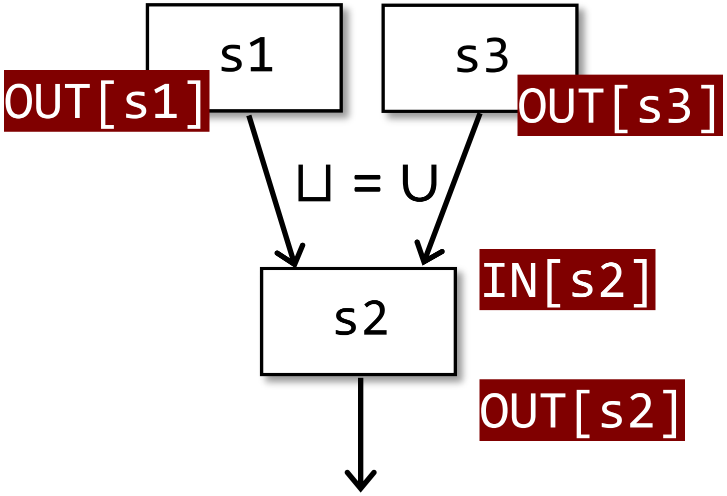

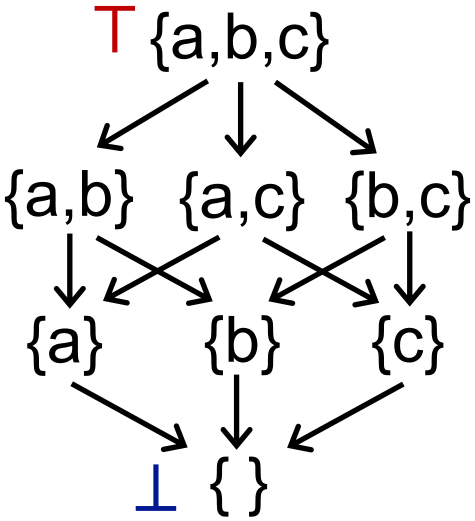

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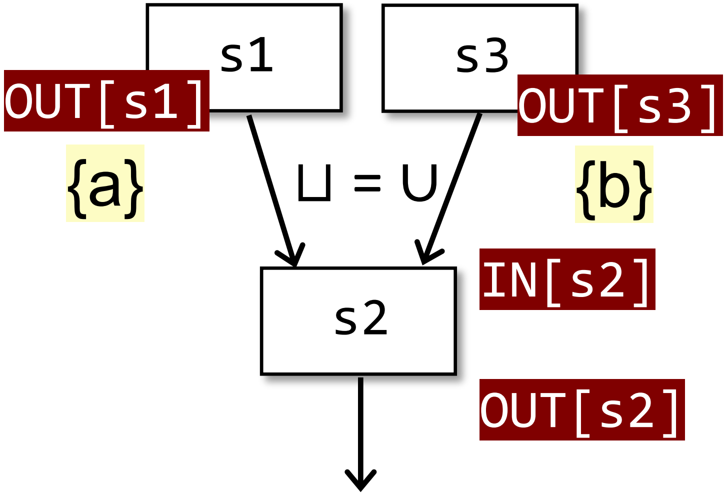

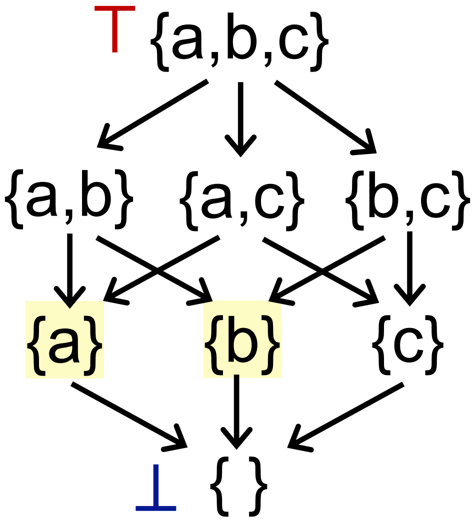

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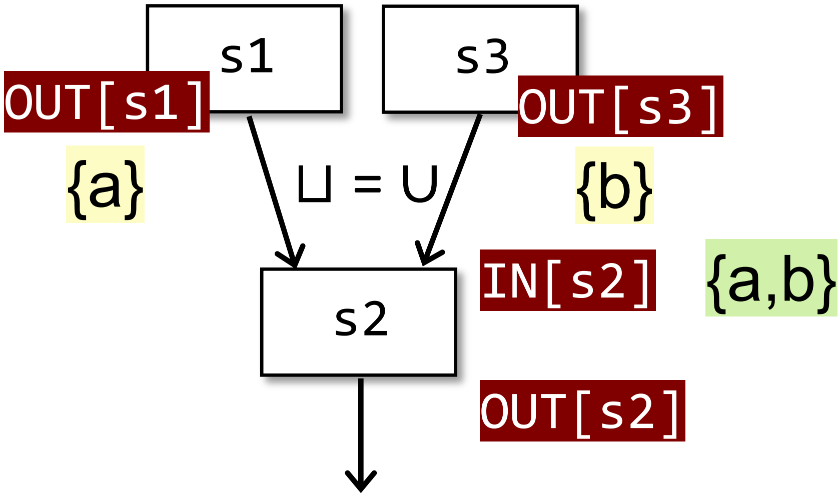

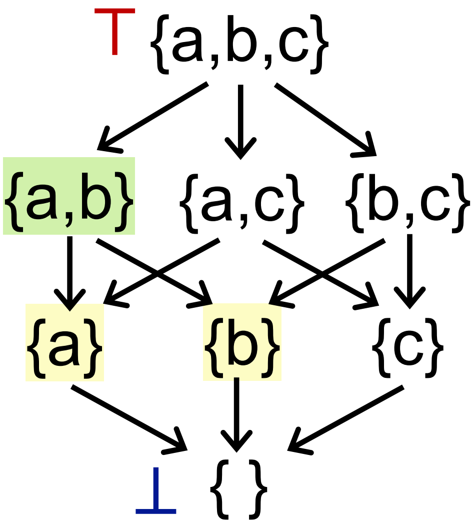

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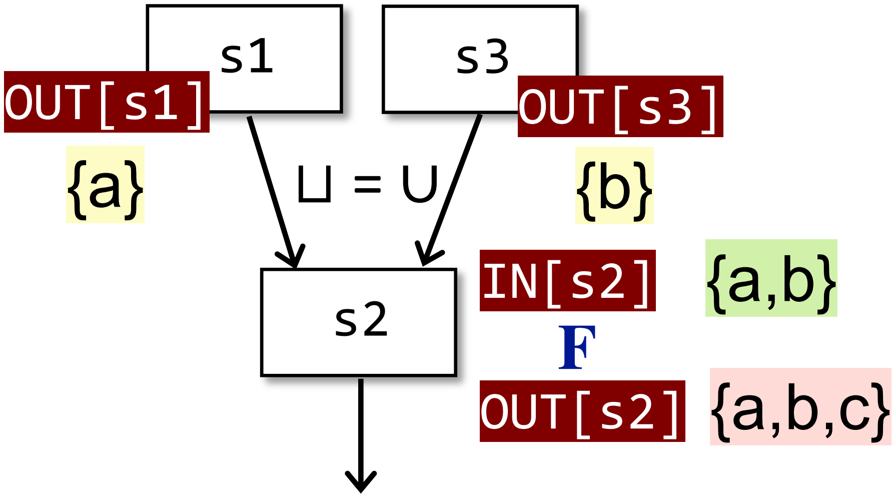

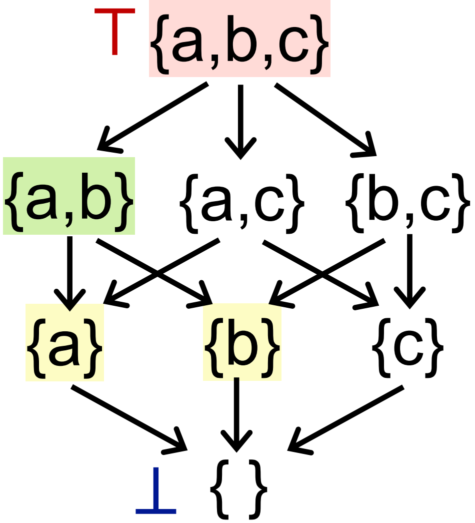

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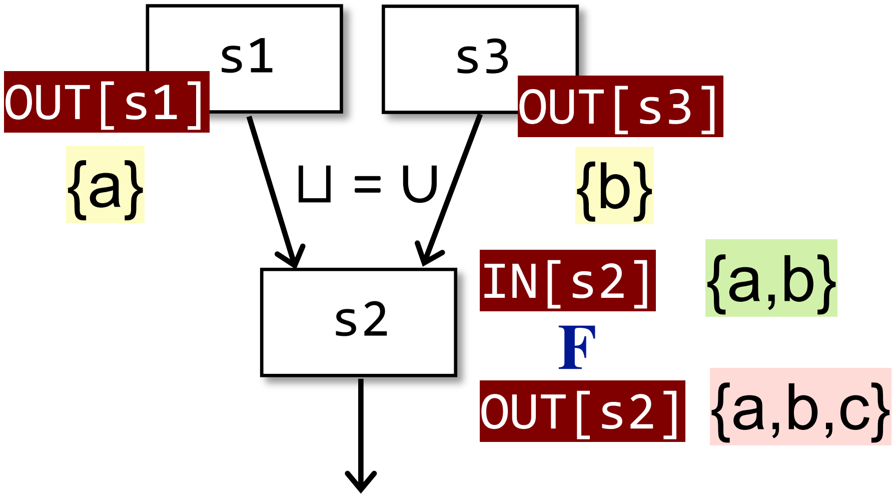

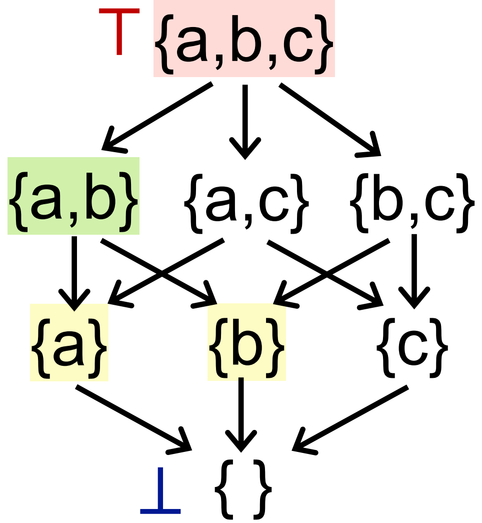

Data flow analysis can be seen as iteratively applying transfer functions and meet/join operations on the values of a lattice

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The iterative algorithm (or the IN/OUT equation system) produces a solution to a data flow analysis

- Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?
- If so, is there only one solution or only one fixed point? If more than one, is our solution the best one (most precise)?
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It is about monotonicity

- ? Is the algorithm guarante minate or reach the fixed point, or does it always have a solution?
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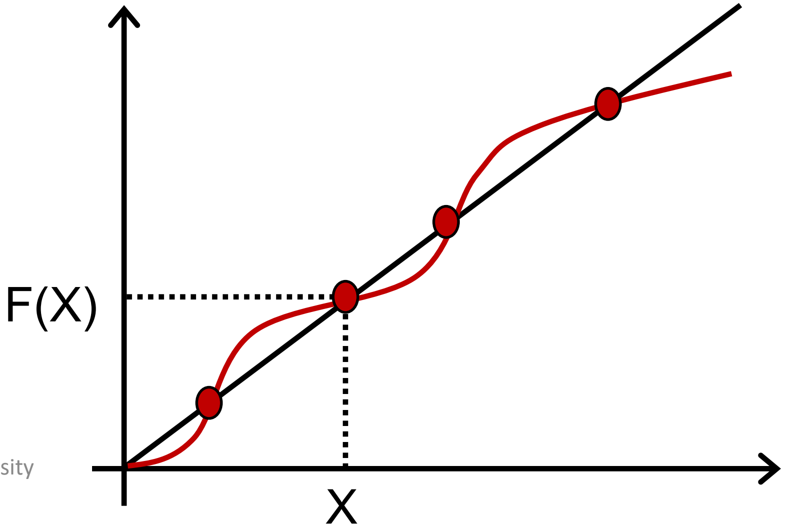

Monotonicity

A function f: L → L (L is a lattice) is monotonic if $\forall x, y \in L$,

$$x \sqsubseteq y \implies f(x) \sqsubseteq f(y)$$

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Let us prove

(1) Existence of fixed point
(2) The fixed point is the least

Fixed-Point Theorem (Existence of Fixed Point)

Proof:

By the definition of ⊥ and f: L → L, we have

$$\bot \sqsubseteq f(\bot)$$

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Similarly (by repeatedly applying f), we have

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As L is finite (and f is monotonic), for some k, we have

$$f^{Fix} = f^k(\perp) = f^{k+1}(\perp)$$

Thus, the fixed point exists.

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Proof:

Assume we have another fixed point x, i.e., x = f(x)

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The proof for greatest fixed point is similar

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Review The Questions We Have Seen Before

The iterative algorithm (or the IN/OUT equation system) produces a solution to a data flow analysis

- ? Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?
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greatest or least fixed point

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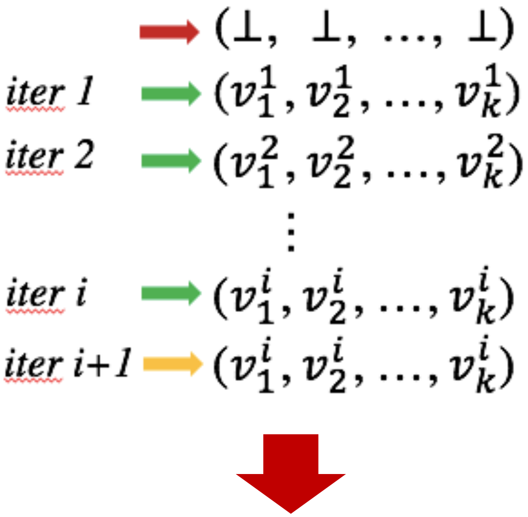
Now what we have just seen is the property (fixed point theorem) for the function on a lattice. We cannot say our iterative algorithm also has that property unless we can *relate the algorithm to the fixed point theorem*, if possible

Relate Iterative Algorithm to Fixed-Point Theorem



$\rightarrow (\bot, \bot, \ldots, \bot)$

iter 1 $\rightarrow (v_1^1, v_2^1, \ldots, v_k^1)$

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iter i $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$

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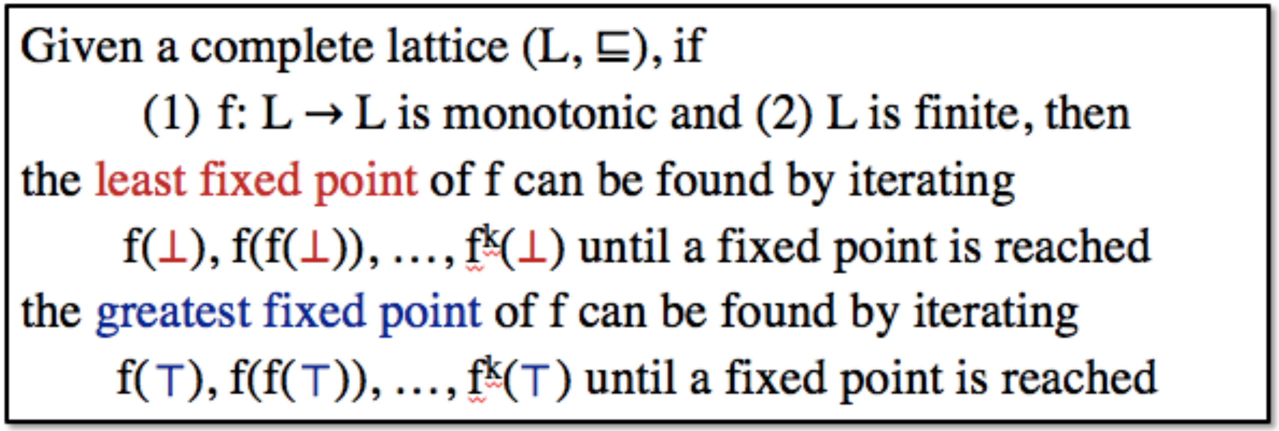

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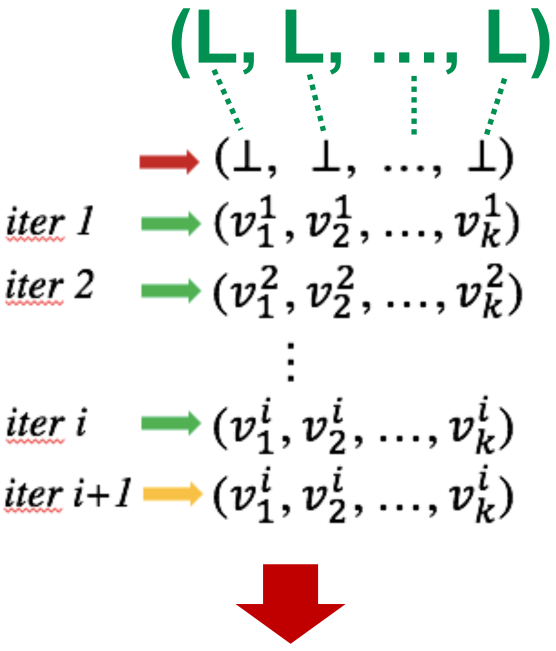
Relate Iterative Algorithm to Fixed-Point Theorem

If a product lattice L^k is a product of complete (and finite) lattices, i.e., (L, L, …, L), then L^k is also complete (and finite)



(L, L, …, L)

$\rightarrow (\bot, \bot, \ldots, \bot)$

iter 1 $\rightarrow (v_1^1, v_2^1, \ldots, v_k^1)$

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⋮

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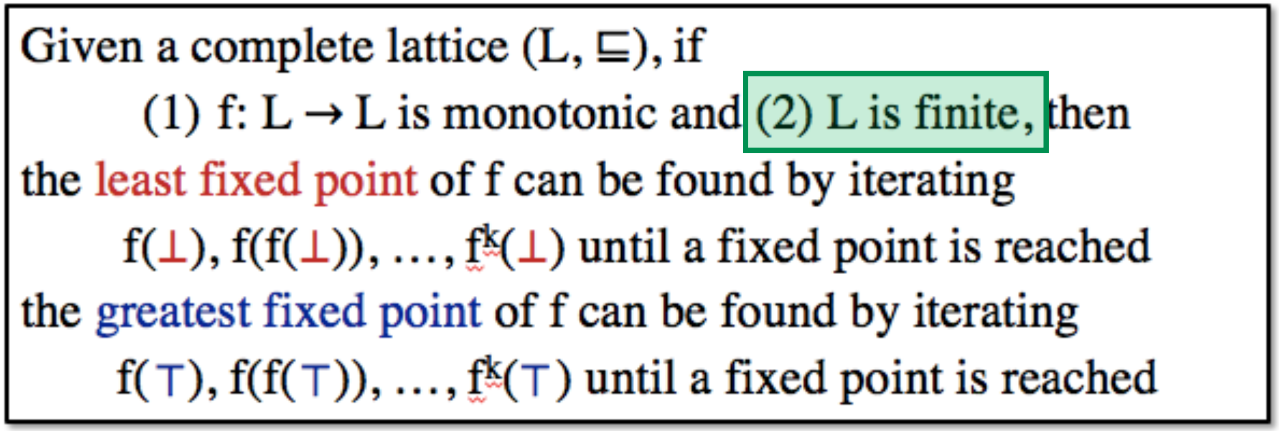

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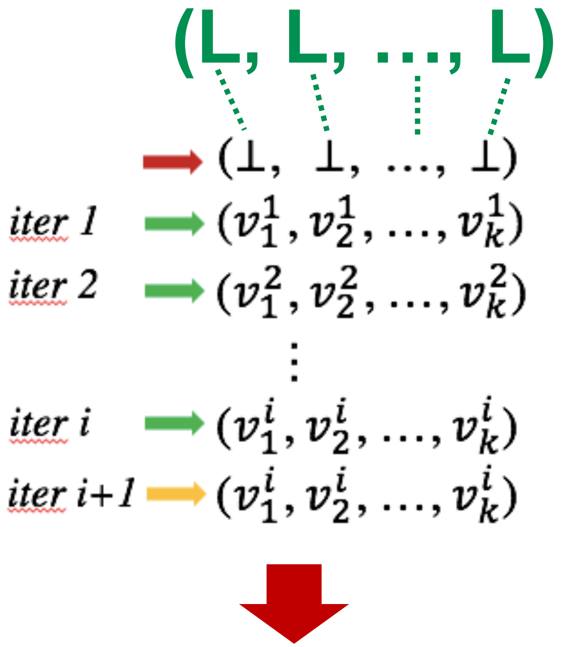

$(L, L, \dots, L)$

$(\perp, \perp, \dots, \perp)$

iter 1 $(v_1^1, v_2^1, \dots, v_k^1)$

iter 2 $(v_1^2, v_2^2, \dots, v_k^2)$

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In each iteration, it is equivalent to think that we apply function F which consists of

(1) transfer function f_i: L → L for every node

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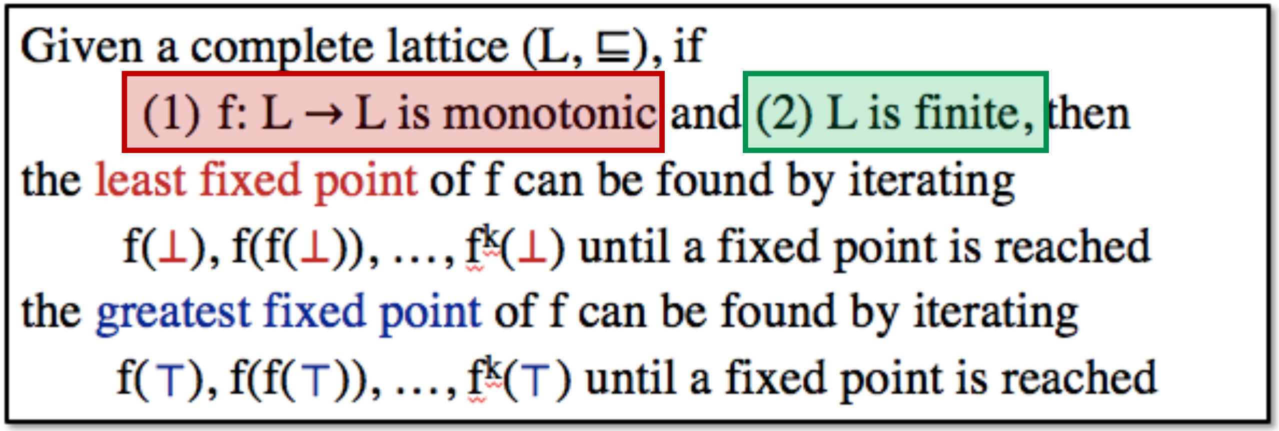

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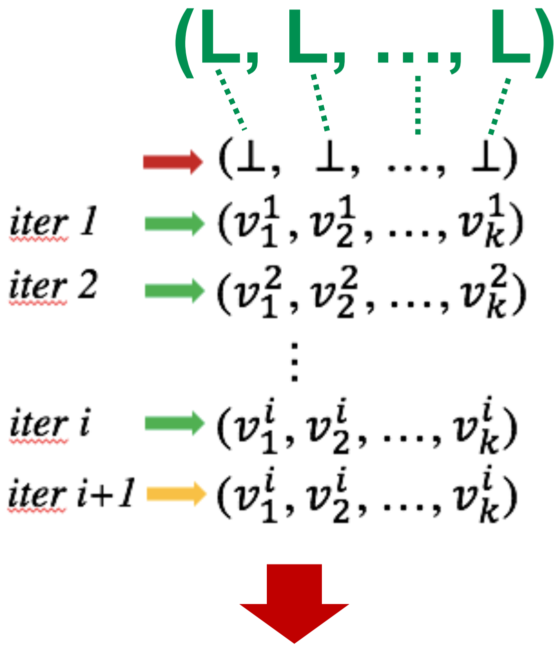

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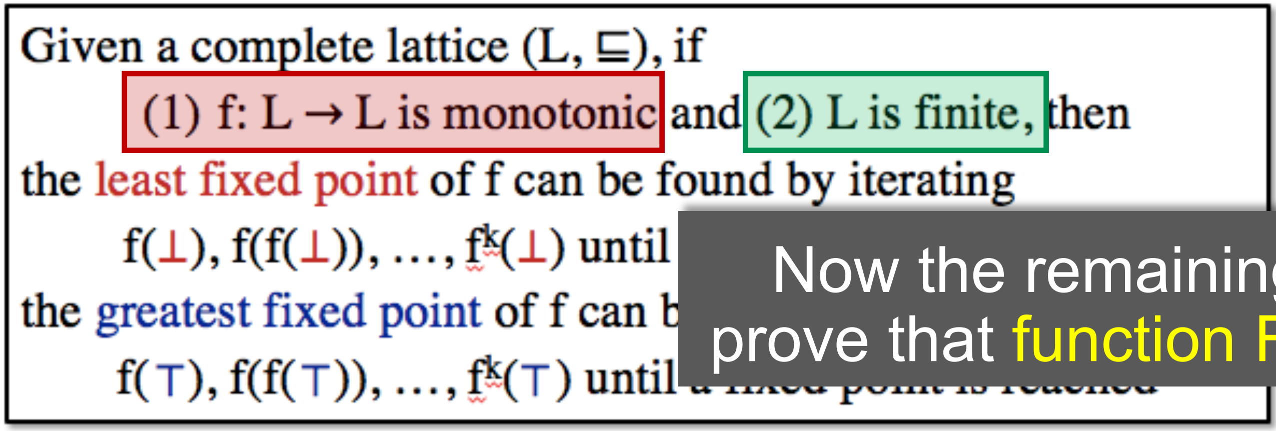

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Now the remaining issue is to prove that function F is monotonic

Prove Function F is Monotonic

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Proof.

$\forall x, y, z \in L$, $x \sqsubseteq y$, we want to prove $x \sqcup z \sqsubseteq y \sqcup z$

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by transitivity of $\sqsubseteq$, $x \sqsubseteq y \sqcup z$

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Thus the fixed point theorem applies to the iterative algorithm for data flow analysis

Review The Questions We Have Seen Before

The iterative algorithm (or the IN/OUT equation system) produces a solution to a data flow analysis

- ? Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?
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greatest or least fixed point

- When will the algorithm reach the fixed point, or when can we get the solution?

Now what we have just seen is the property (fixed point theorem) for the function on a lattice. ~~We cannot say our iterative algorithm also has that property unless~~ we can relate the algorithm to the fixed point theorem, ~~if possible~~

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When Will the Algorithm Reach the Fixed Point?

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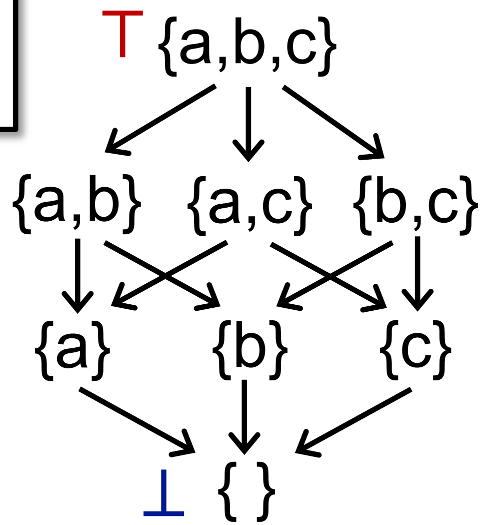
The height of a lattice *h* is the length of the longest path from Top to Bottom in the lattice.



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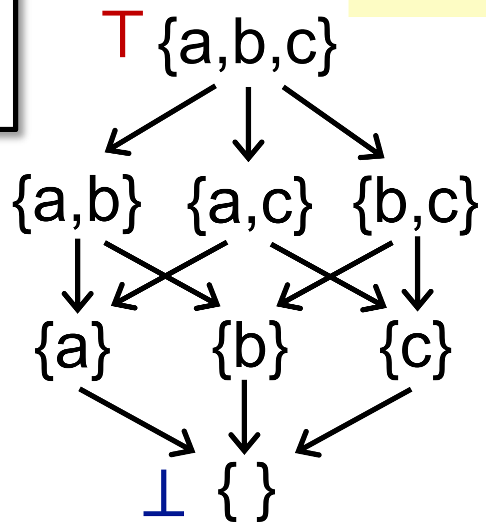

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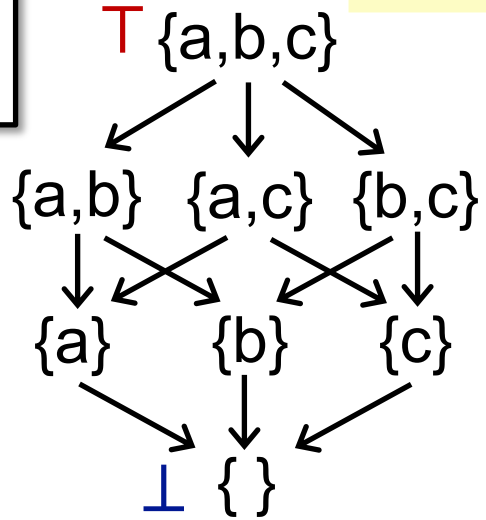

The maximum iterations i needed to reach the fixed point

$\rightarrow (\bot, \bot, \dots, \bot)$

iter 1 $\rightarrow (v_1^1, v_2^1, \dots, v_k^1)$

iter 2 $\rightarrow (v_1^2, v_2^2, \dots, v_k^2)$

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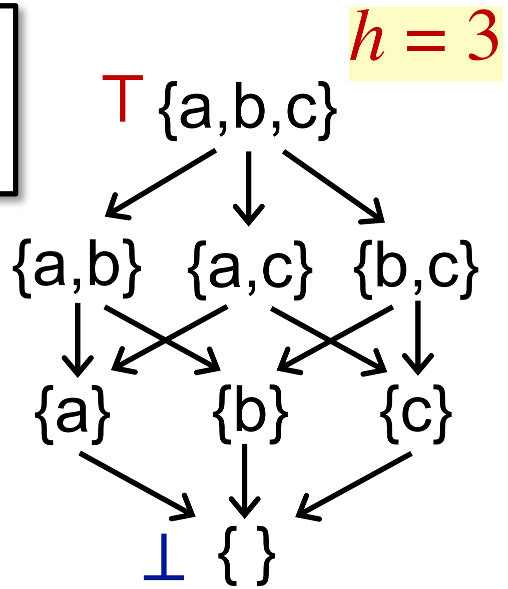

The maximum iterations i needed to reach the fixed point

$$\rightarrow (\bot,\ \bot,\ \ldots,\ \bot)$$

iter 1 $\rightarrow (v_1^1, v_2^1, \ldots, v_k^1)$

iter 2 $\rightarrow (v_1^2, v_2^2, \ldots, v_k^2)$

$\vdots$

iter i $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$

iter i+1 $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$

In each iteration, assume only one step in the lattice (upwards or downwards) is made in one node (e.g., one 0->1 in RD)

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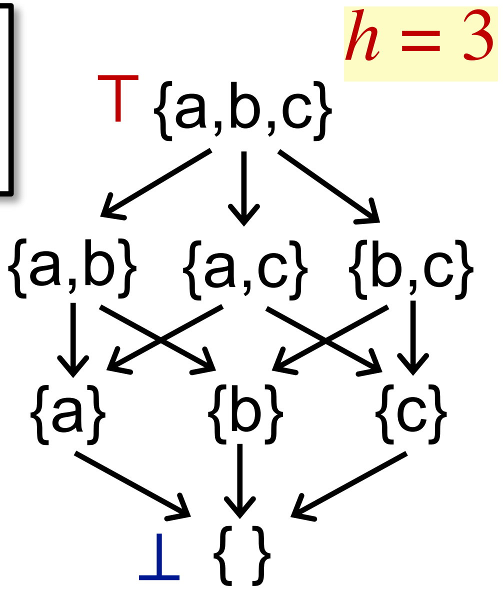

The maximum iterations i needed to reach the fixed point

$\rightarrow (\perp, \perp, \ldots, \perp)$

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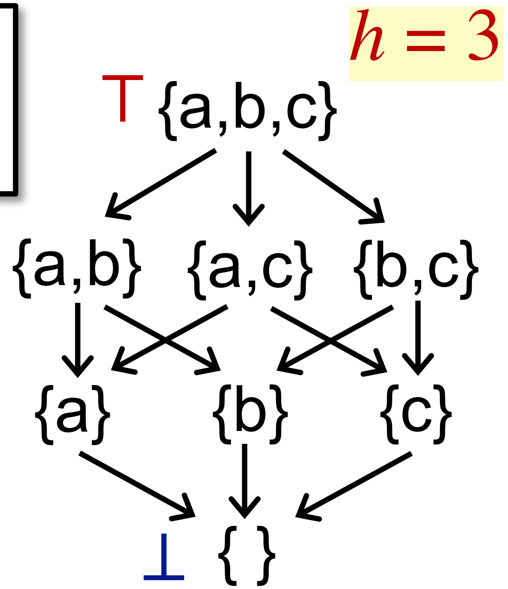

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In each iteration, assume only one step in the lattice (upwards or downwards) is made in one node (e.g., one 0->1 in RD)

Assume the lattice height is h and the number of nodes in CFG is k

We need at most $i = h * k$ iterations

Review The Questions We Have Seen Before

The iterative algorithm (or the IN/OUT equation system) produces a solution to a data flow analysis

YES

✓
- Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?

✓
- If so, ~~is there only one solution or only one fixed point~~? If more than one, is our solution the best one (most precise)?

YES

?
- When will the algorithm reach the fixed point, or when can we get the solution?

Review The Questions We Have Seen Before

The iterative algorithm (or the IN/OUT equation system) produces a solution to a data flow analysis

YES

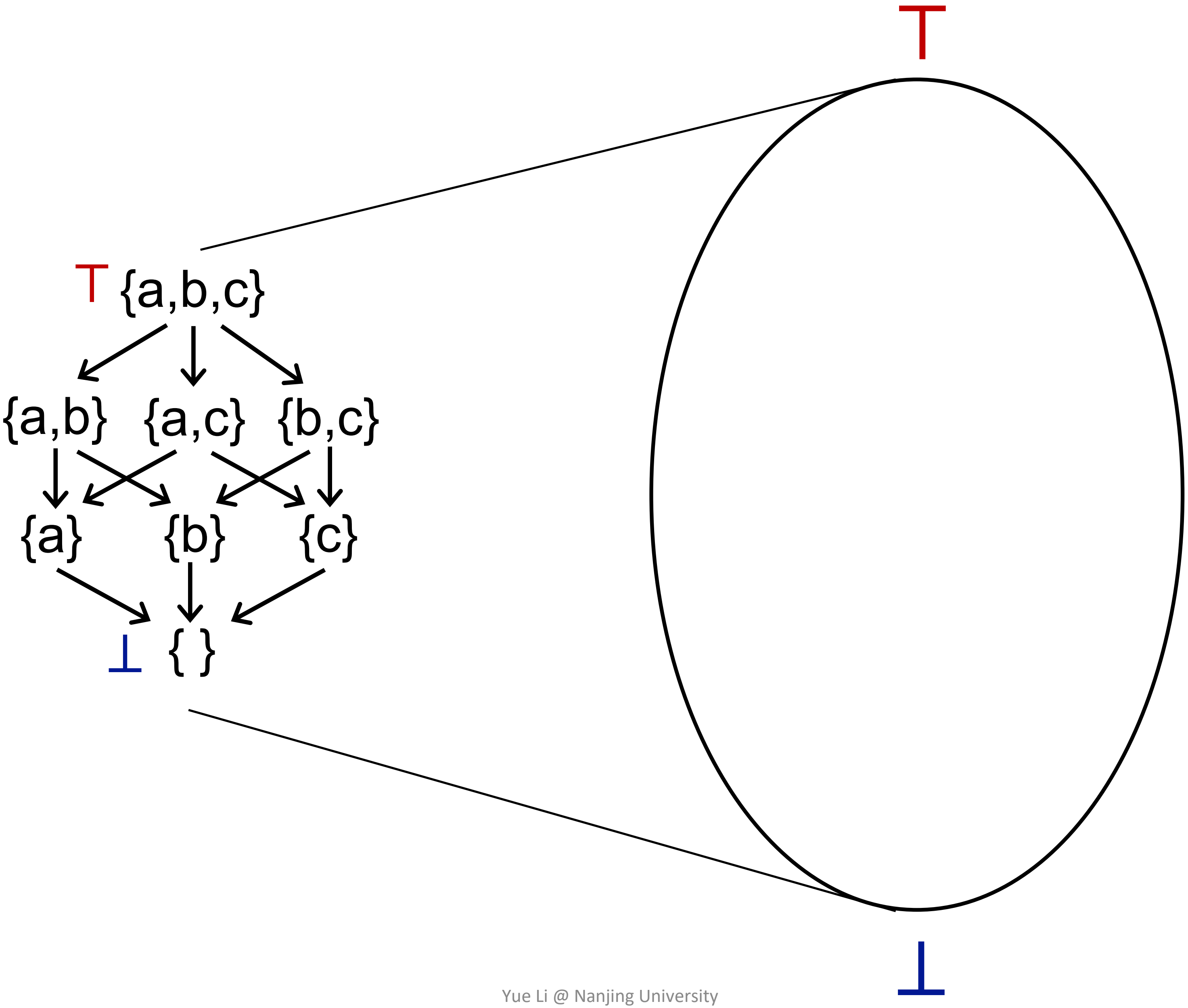
- ✓ Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?
- ✓ If so, ~~is there only one solution or only one fixed point~~? If more than one, is our solution the best one (most precise)?

YES

- ✓ When will the algorithm reach the fixed point, or when can we get the solution?

Worst case of #iterations:
the product of the lattice height and
the number of nodes in CFG

May and Must Analyses, a Lattice View



⊤

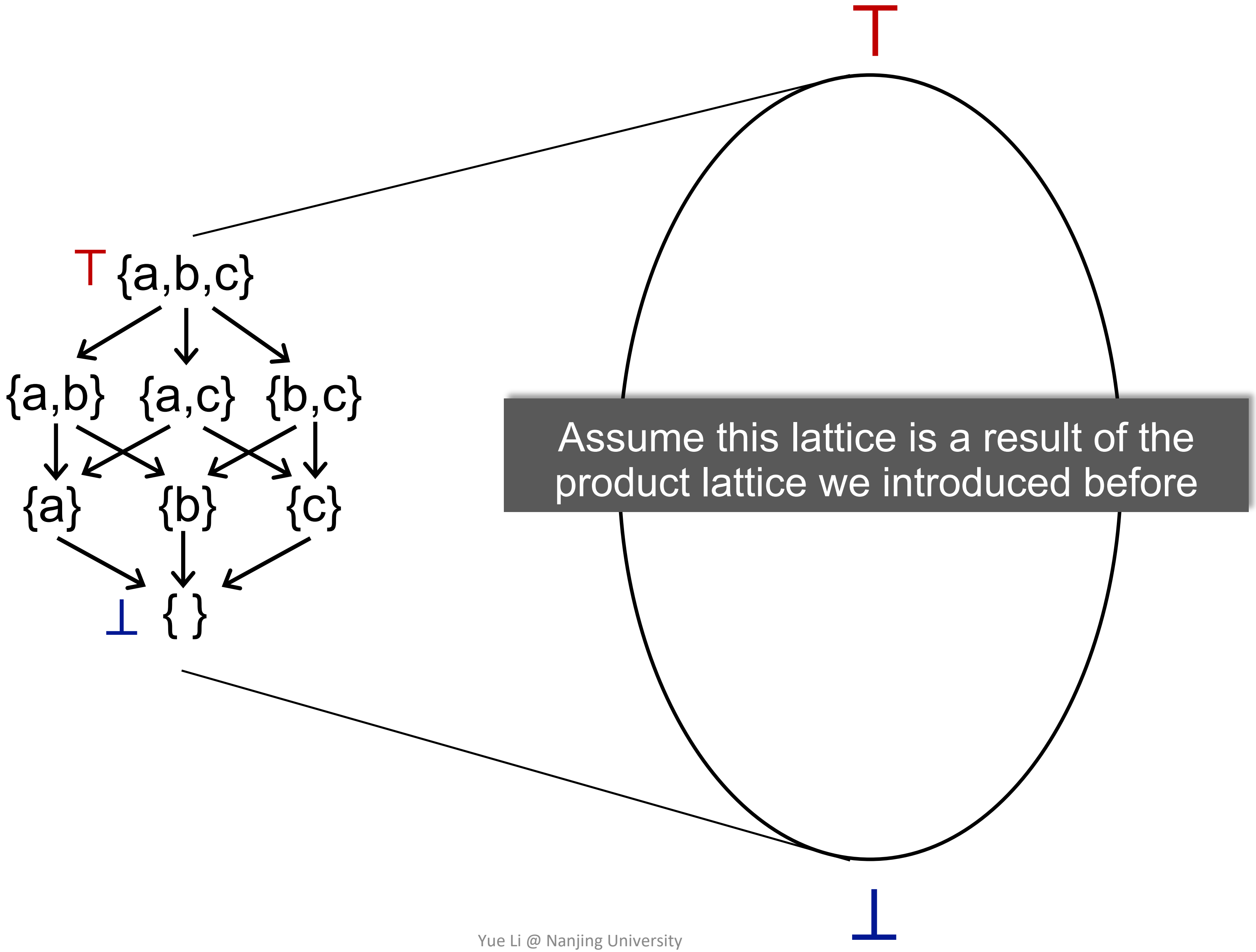
⊤ {a,b,c}

{a,b} {a,c} {b,c}

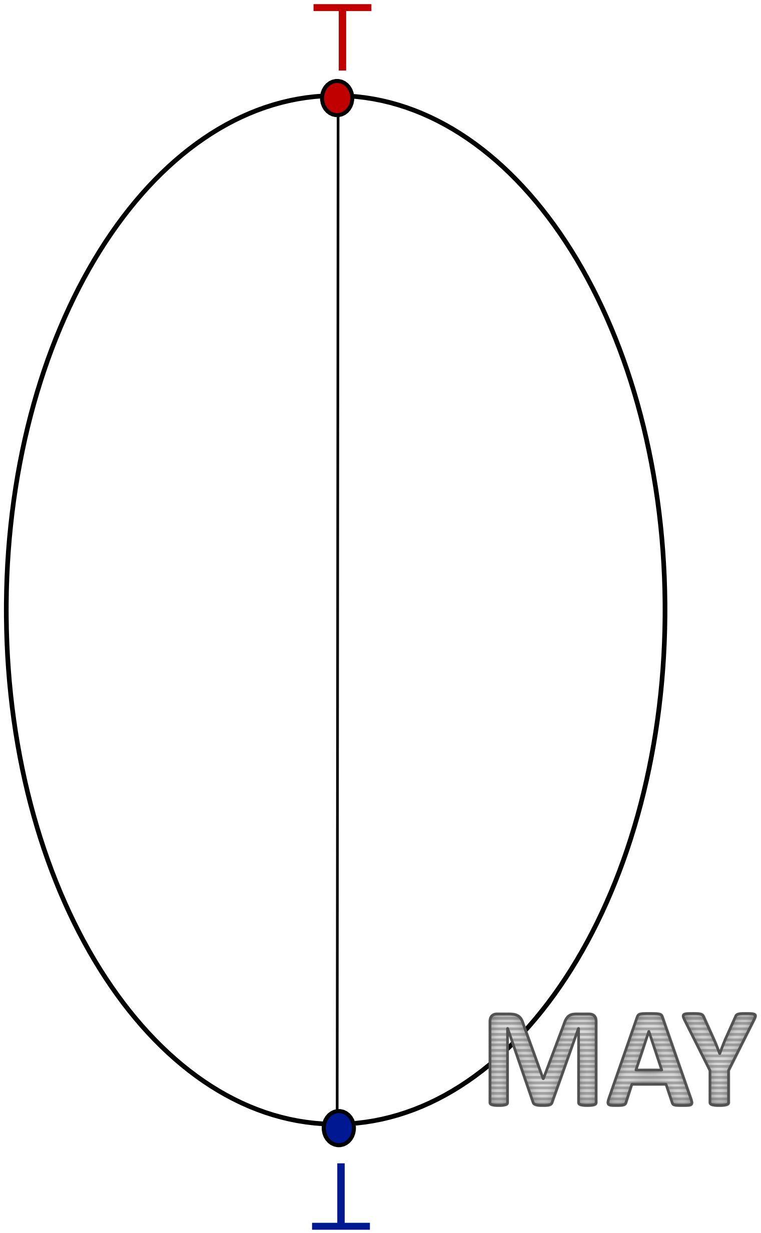
{a} {b} {c}

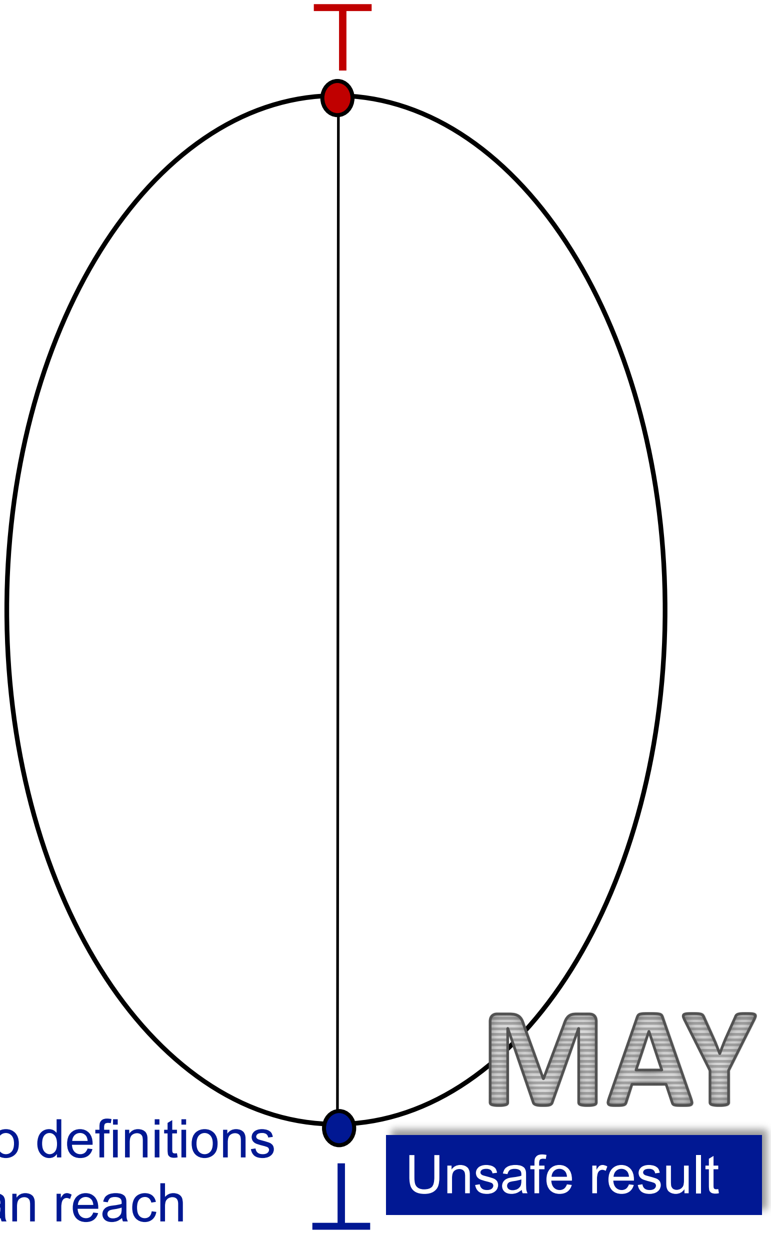
⊥ { }

⊥



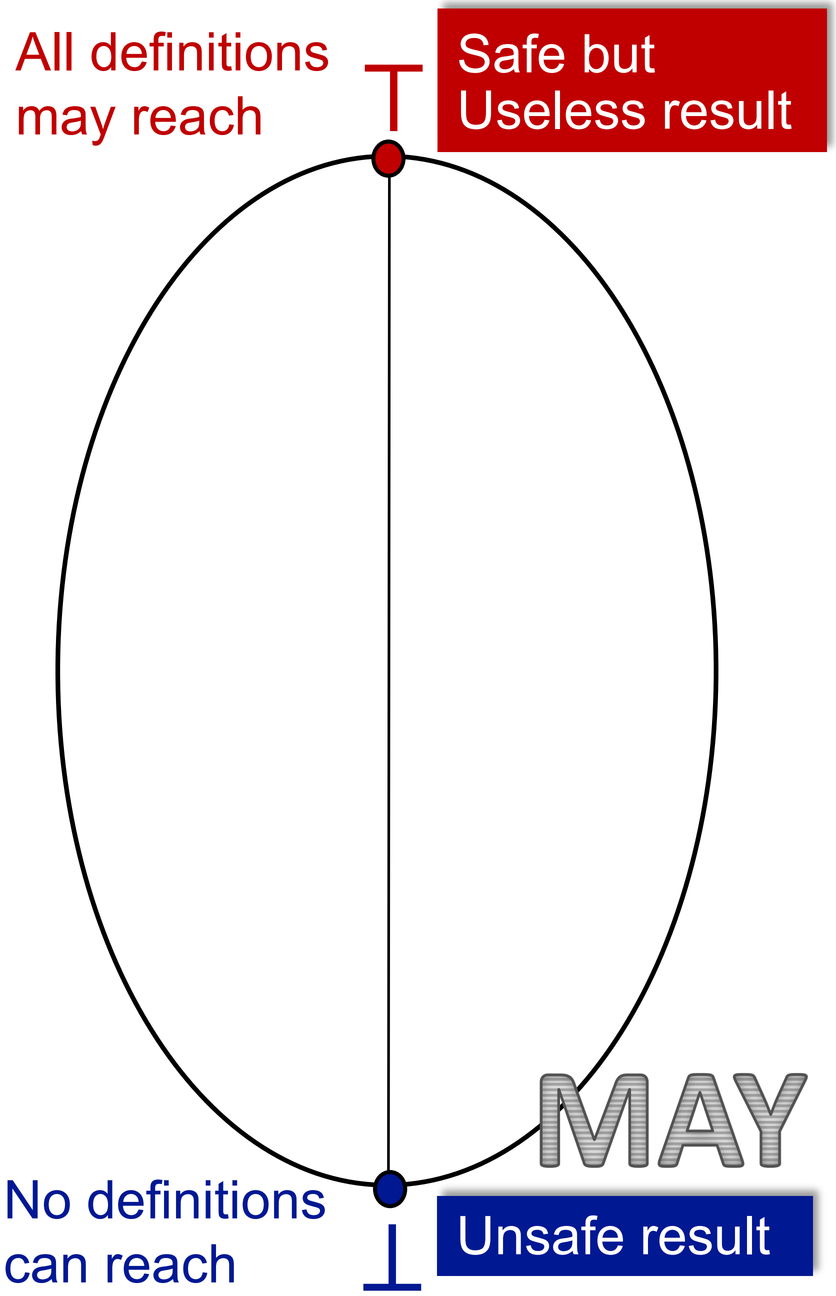
Assume this lattice is a result of the product lattice we introduced before


MAY



No definitions can reach

Unsafe result



All definitions may reach

⊤

Safe but Useless result

MAY

No definitions can reach

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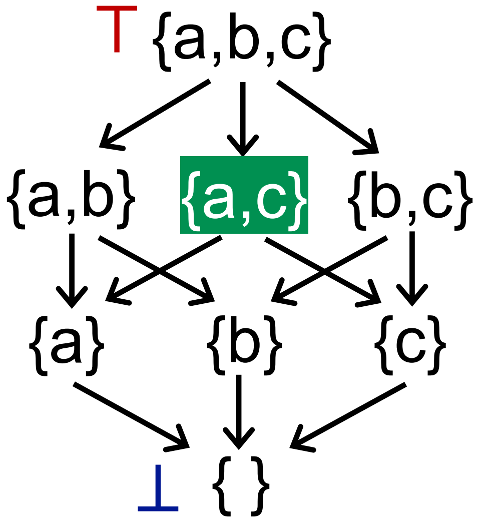
Unsafe result



⊤ {a,b,c}

{a,b} {a,c} {b,c}

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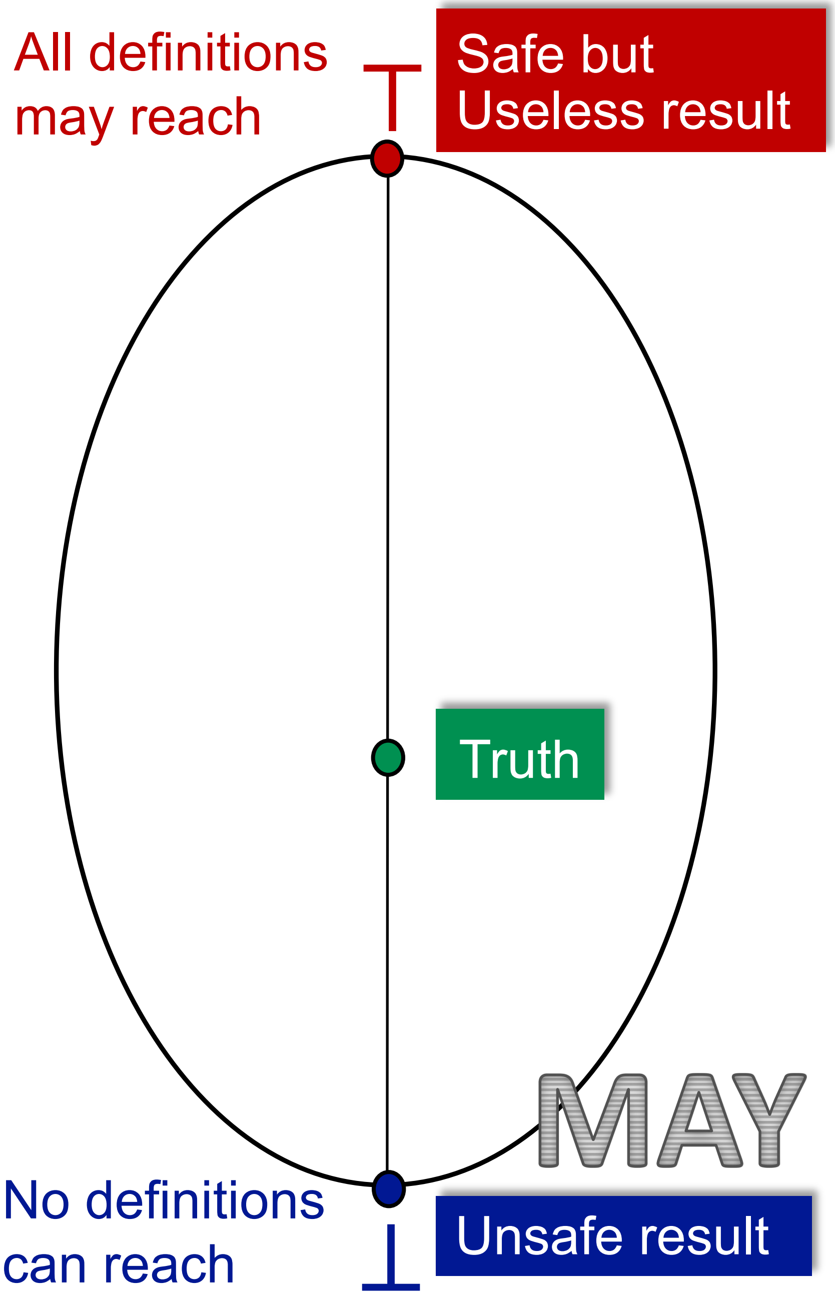

All definitions may reach

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Safe but Useless result

Truth

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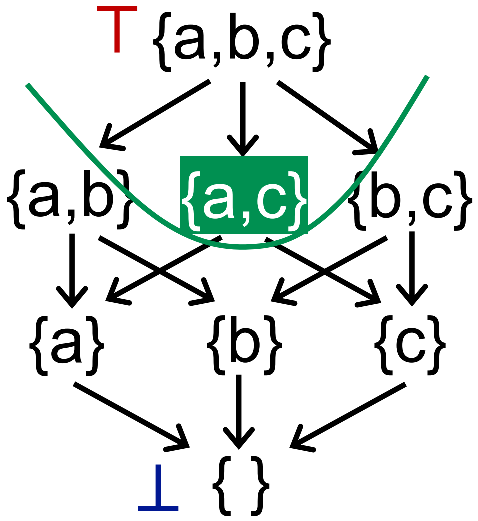
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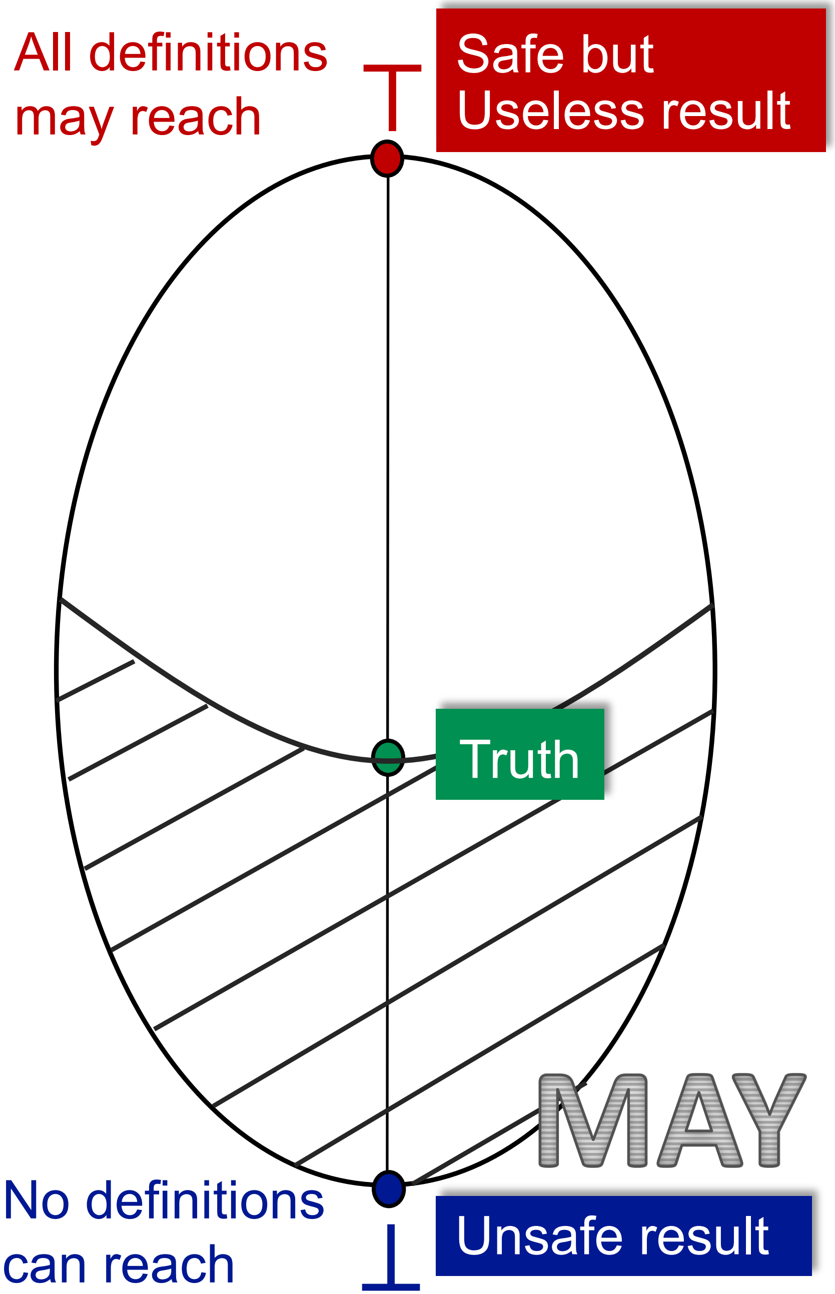
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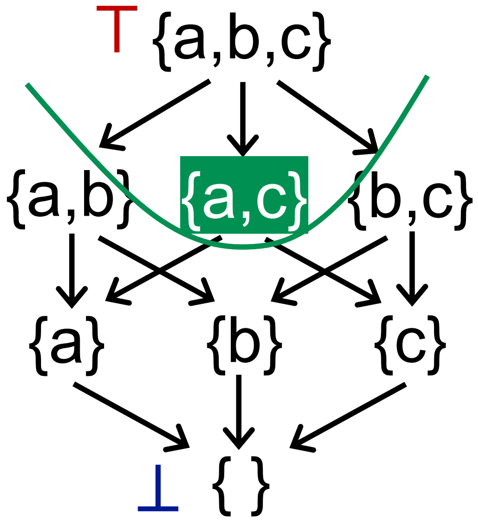

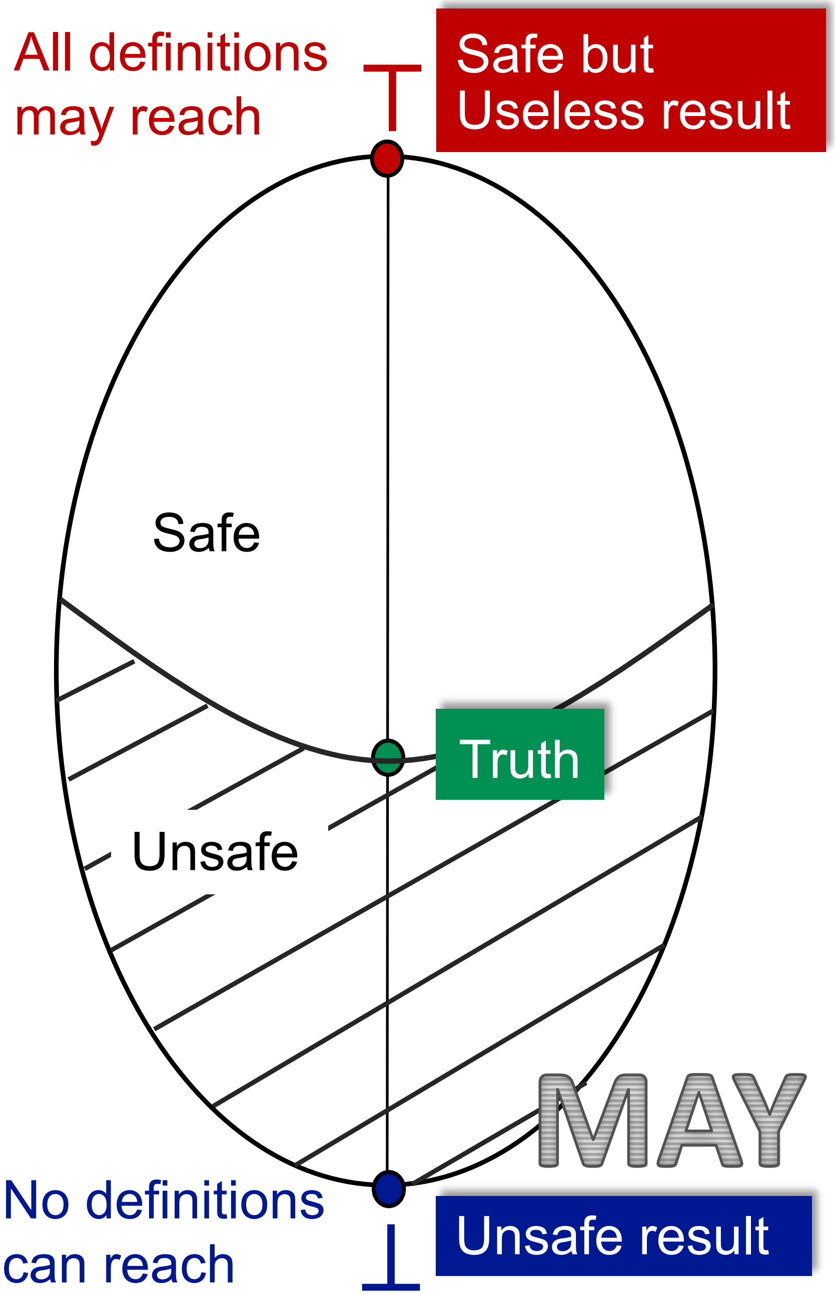

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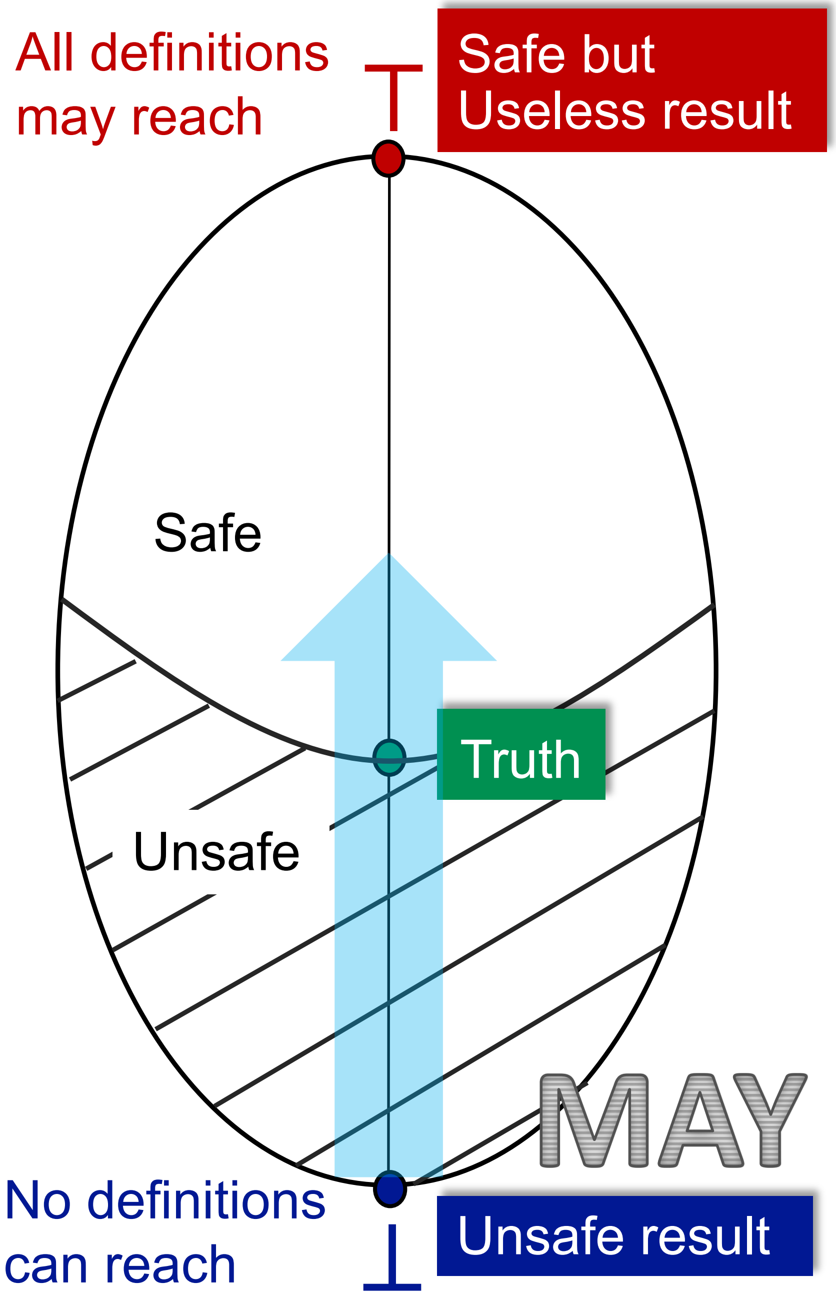

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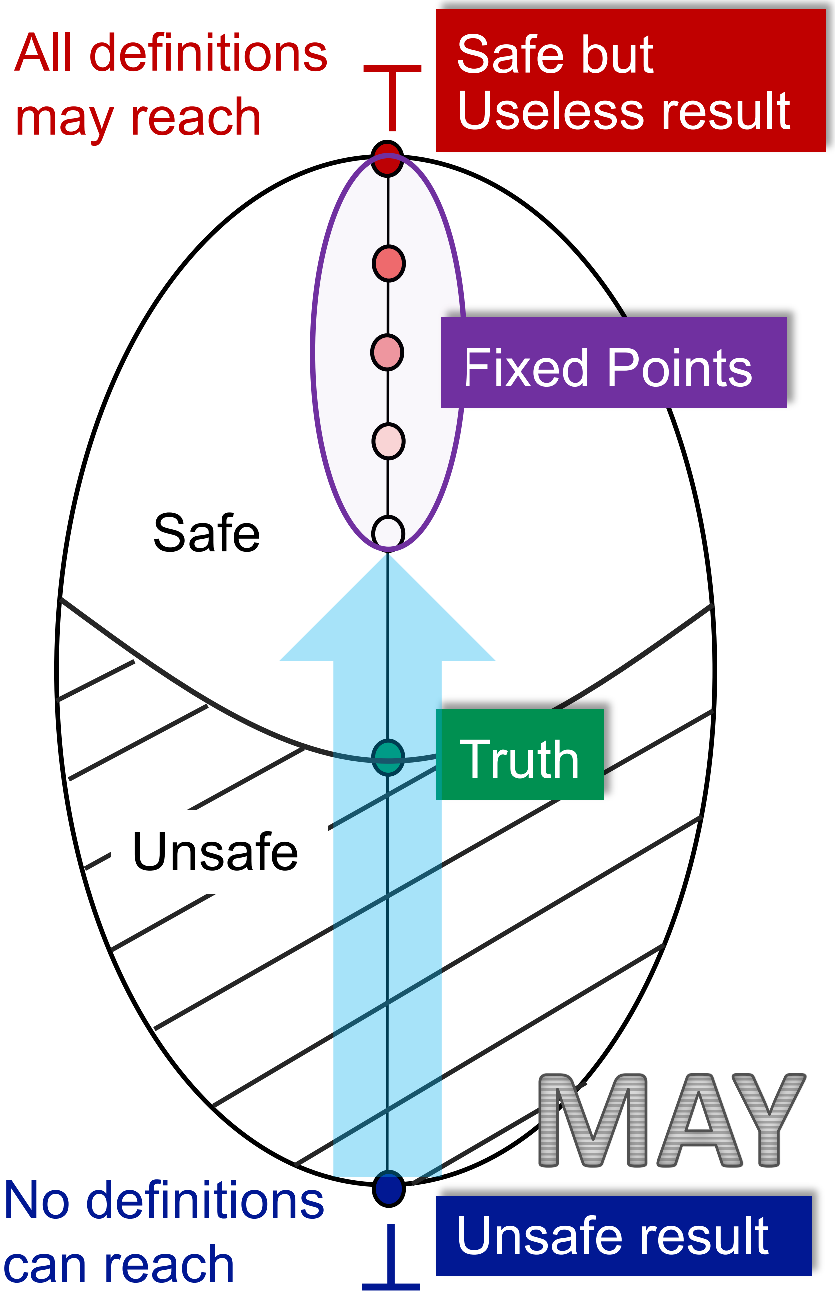

All definitions may reach

Safe but Useless result

Fixed Points

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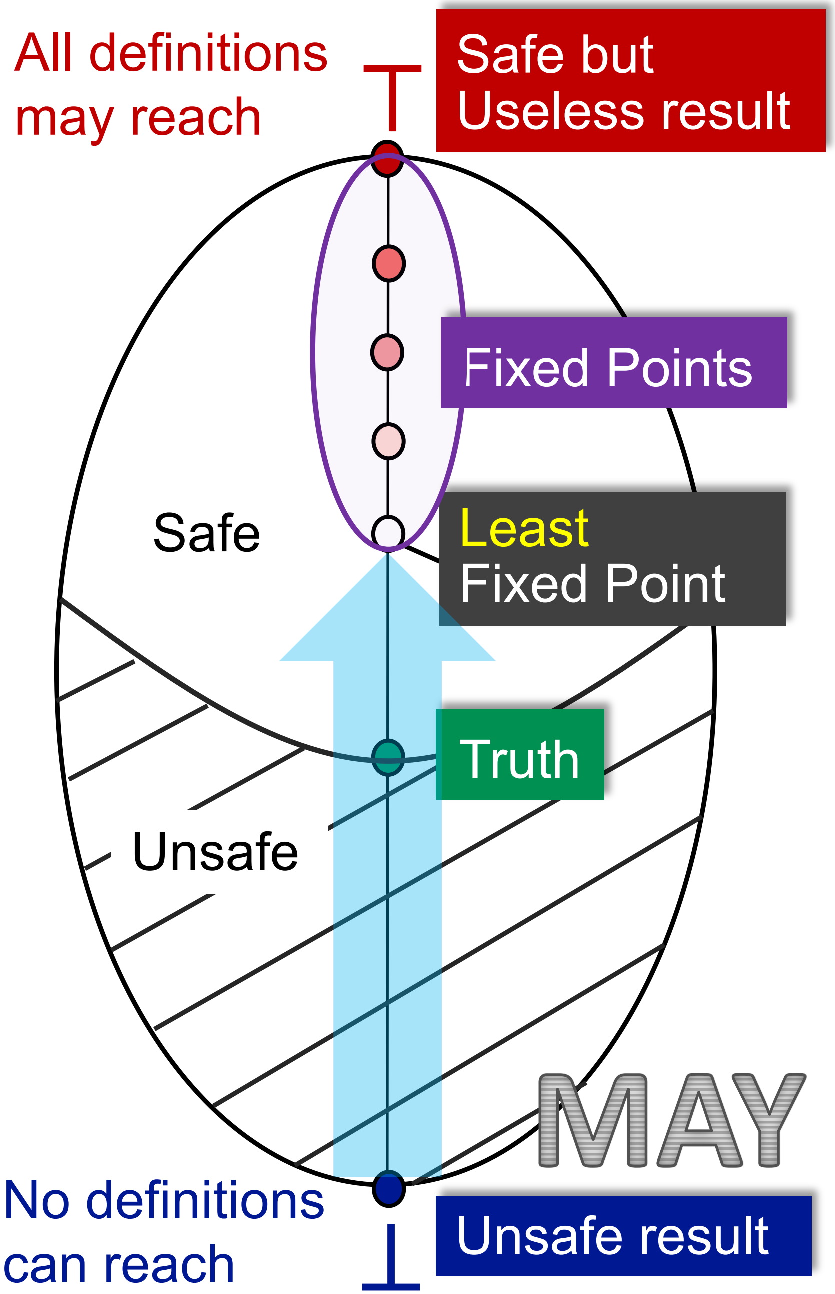
Unsafe result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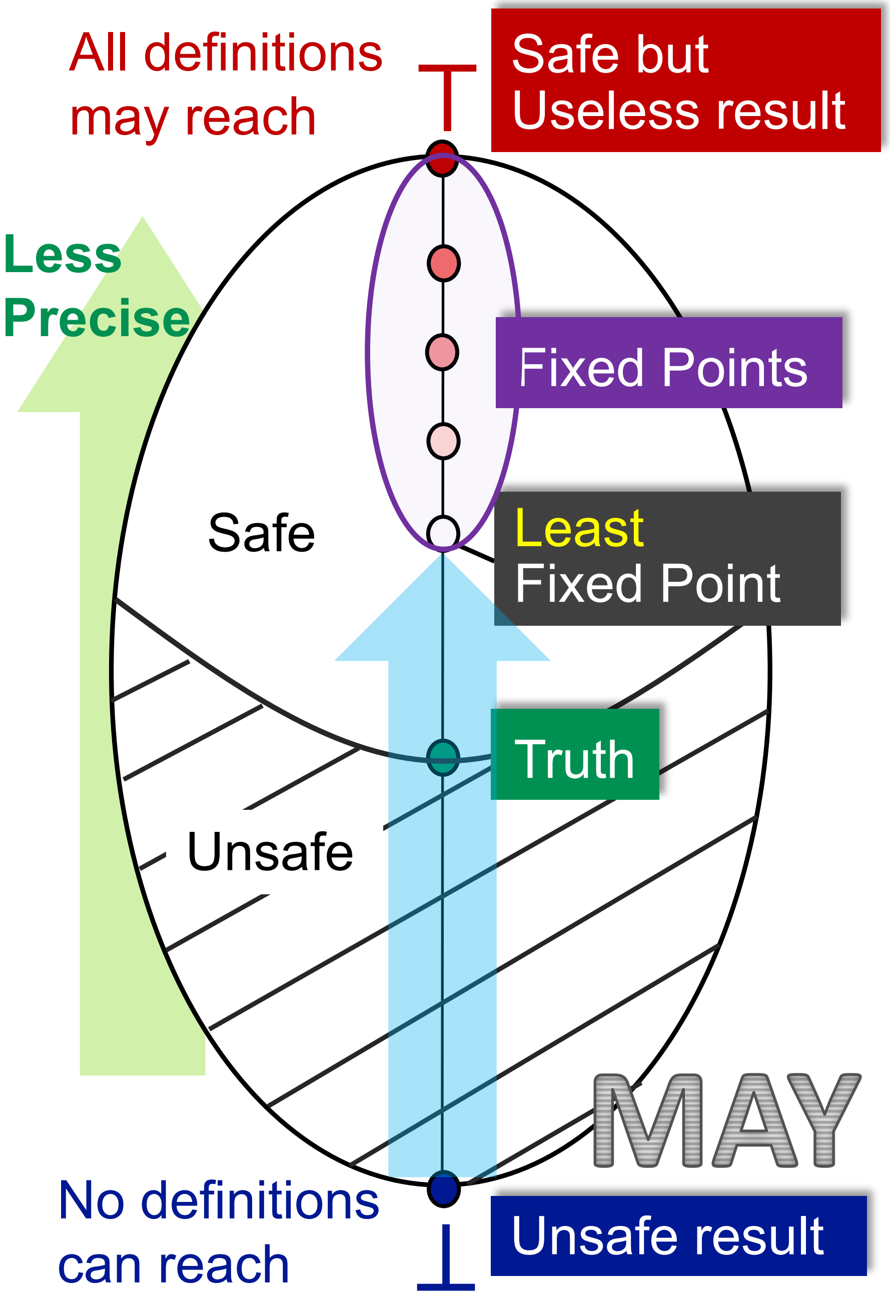
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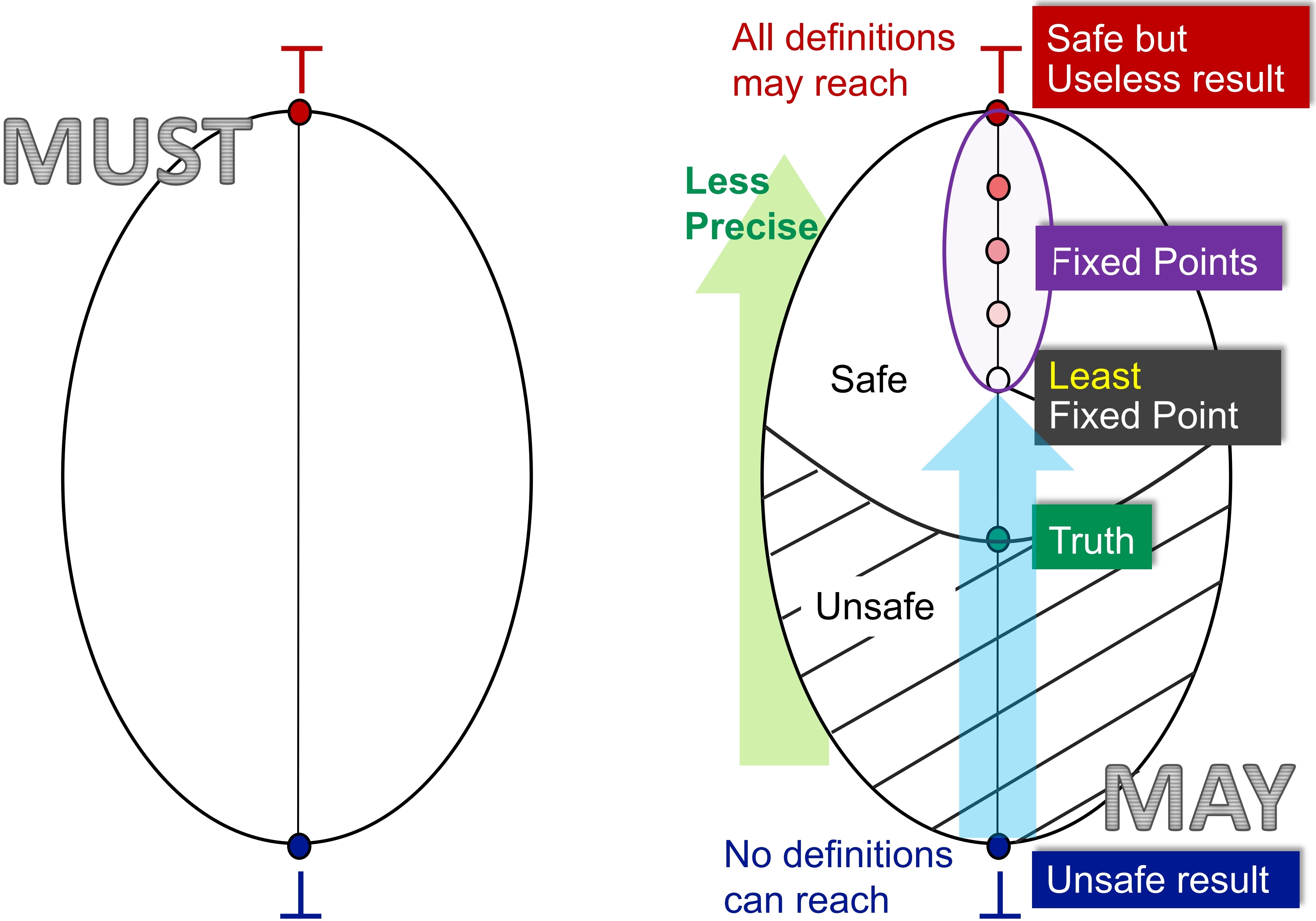
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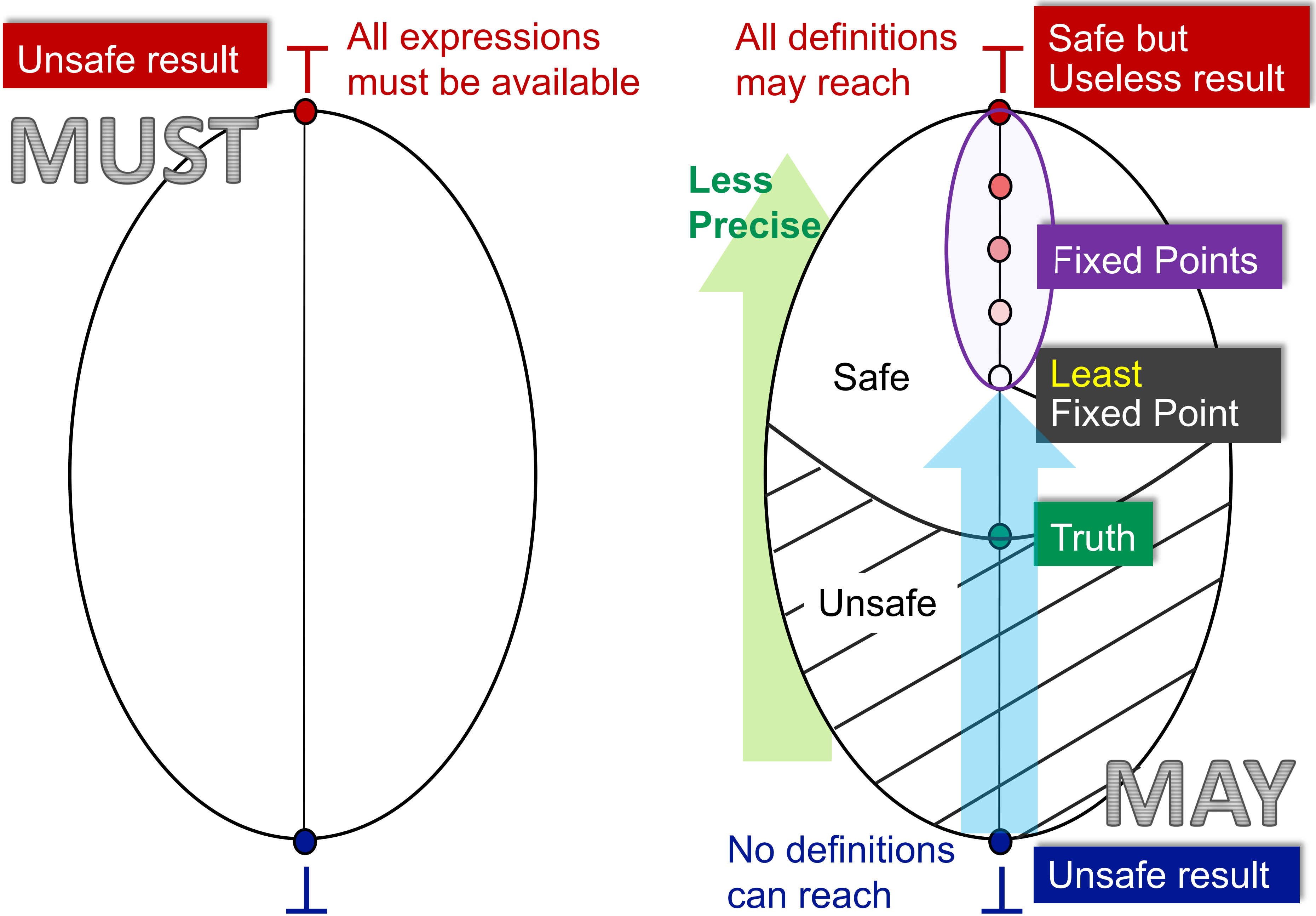

Unsafe result

⊤ All expressions must be available

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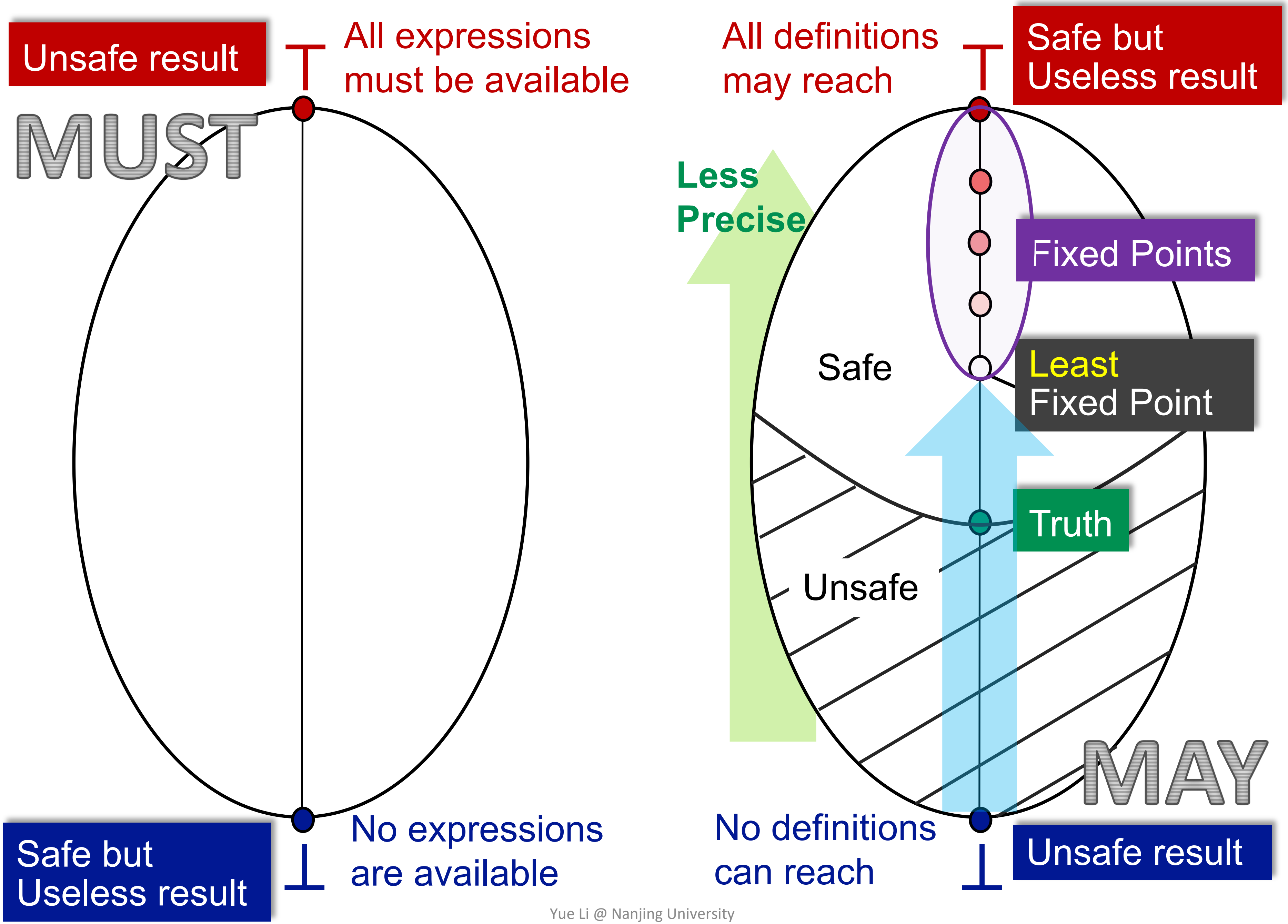

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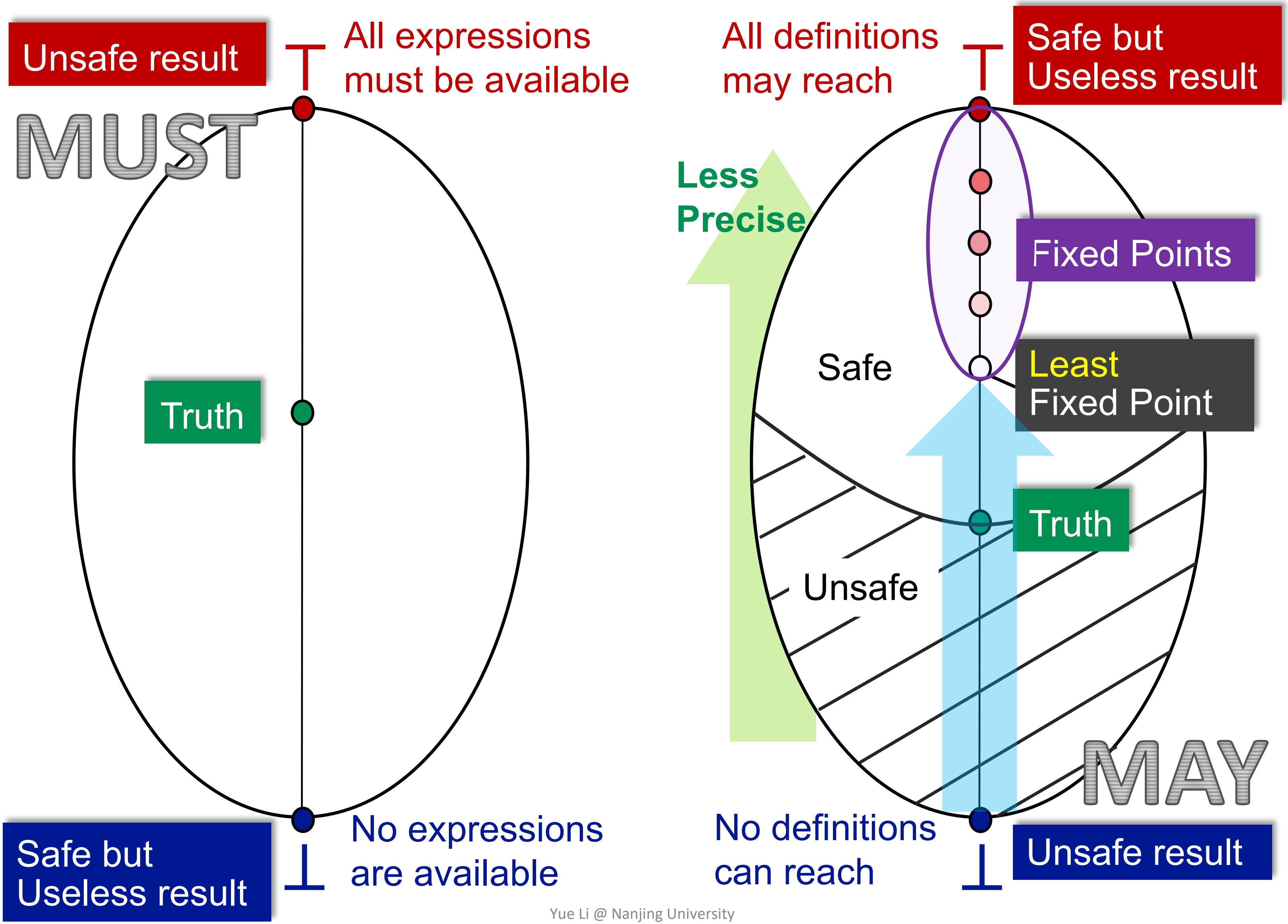

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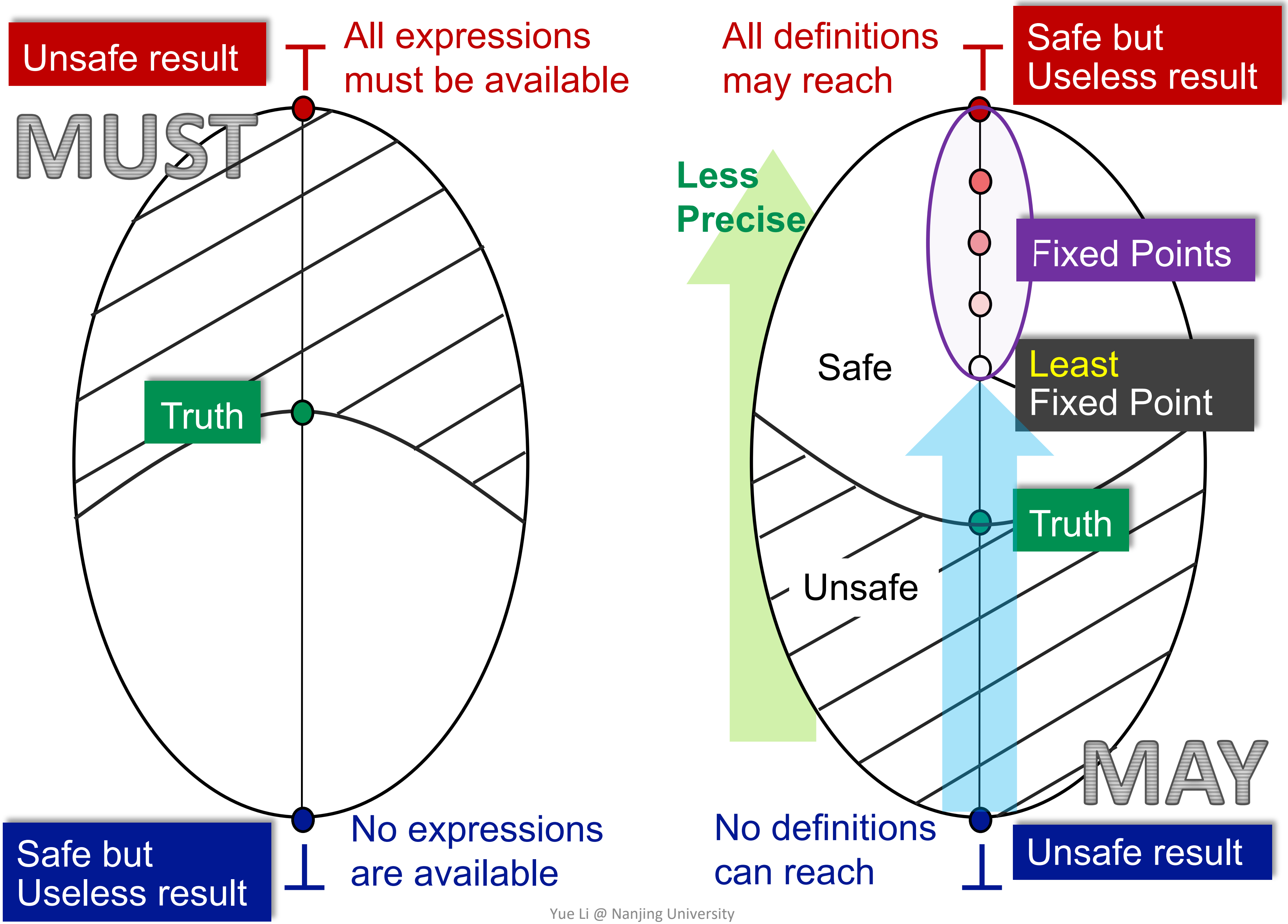

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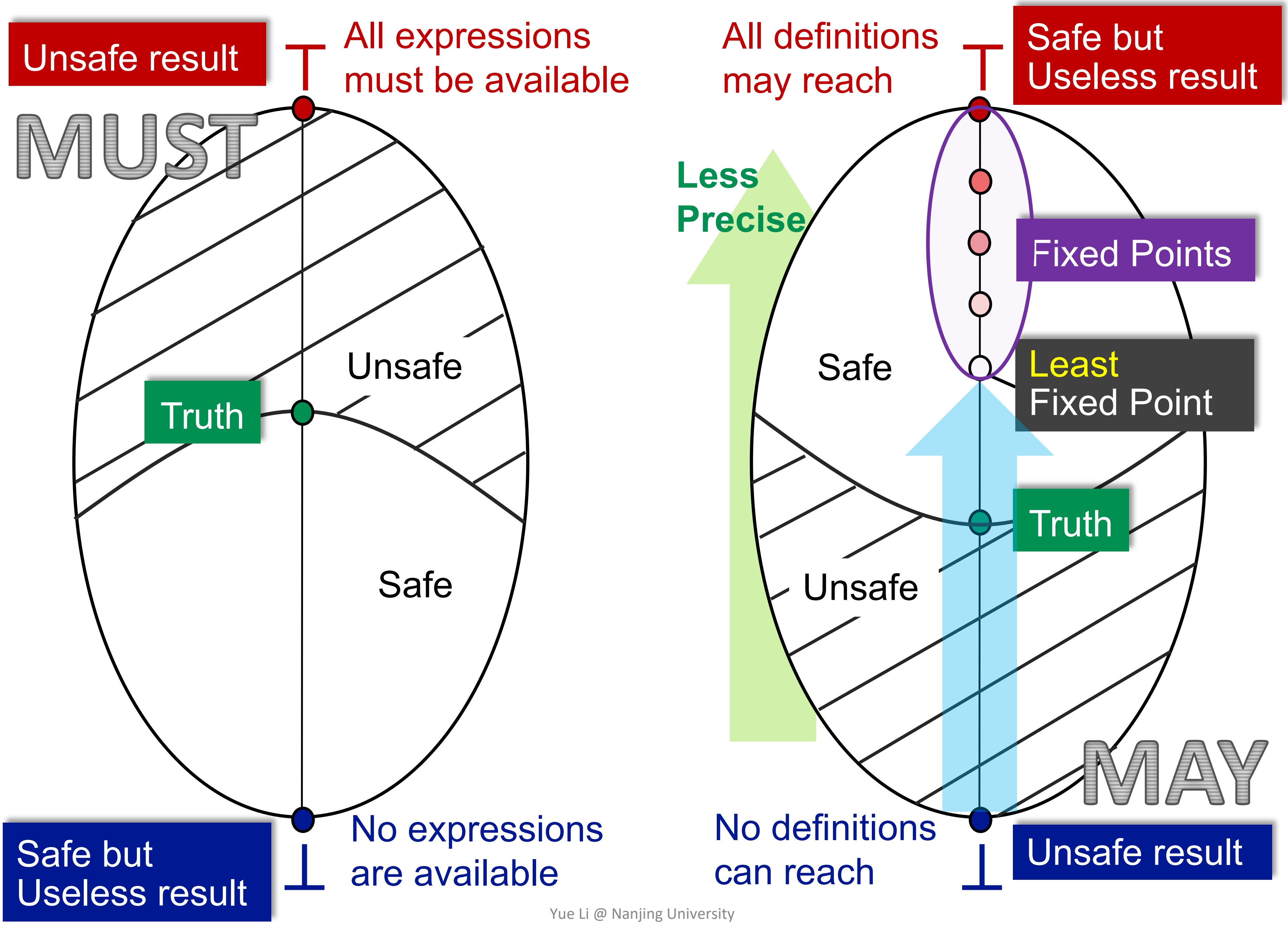

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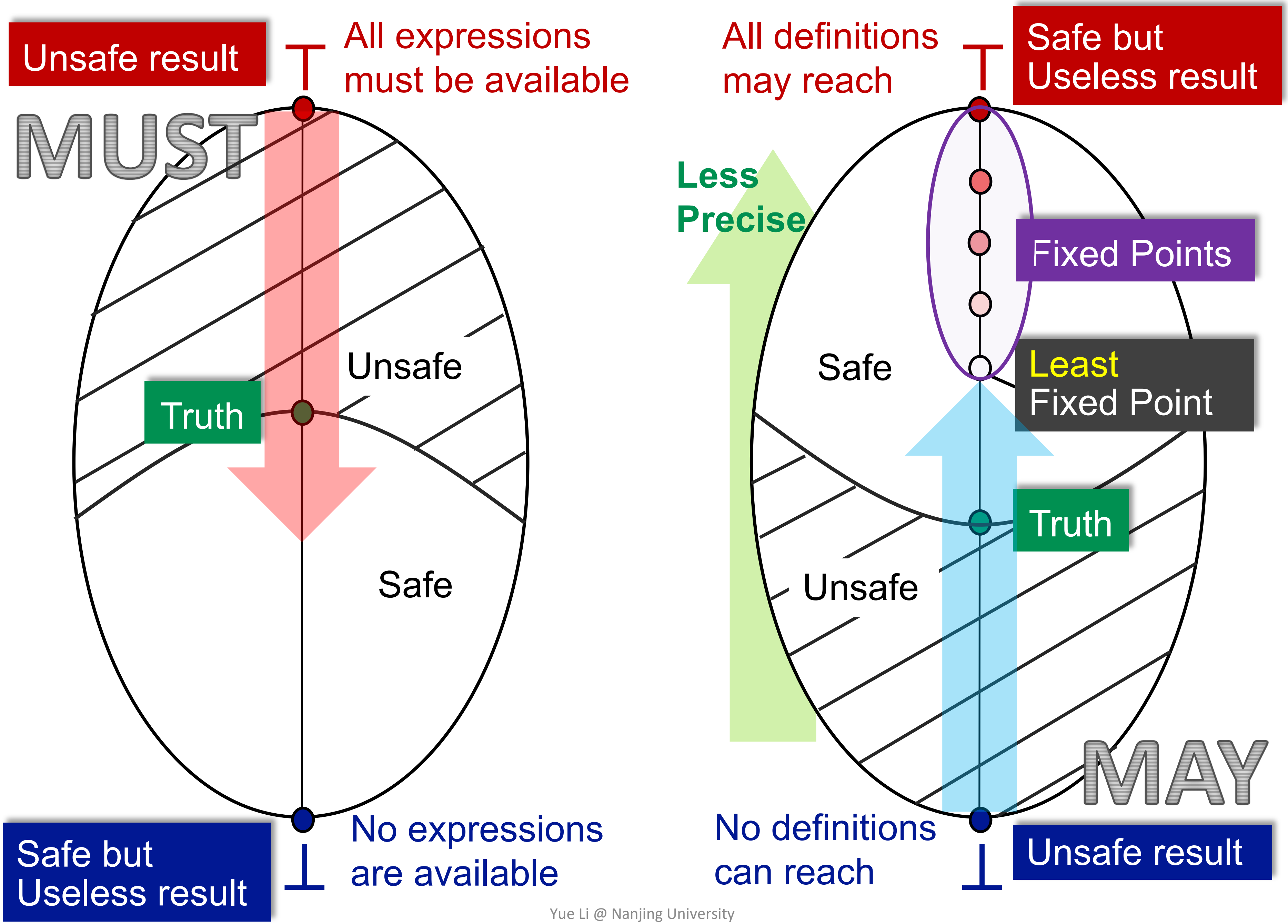

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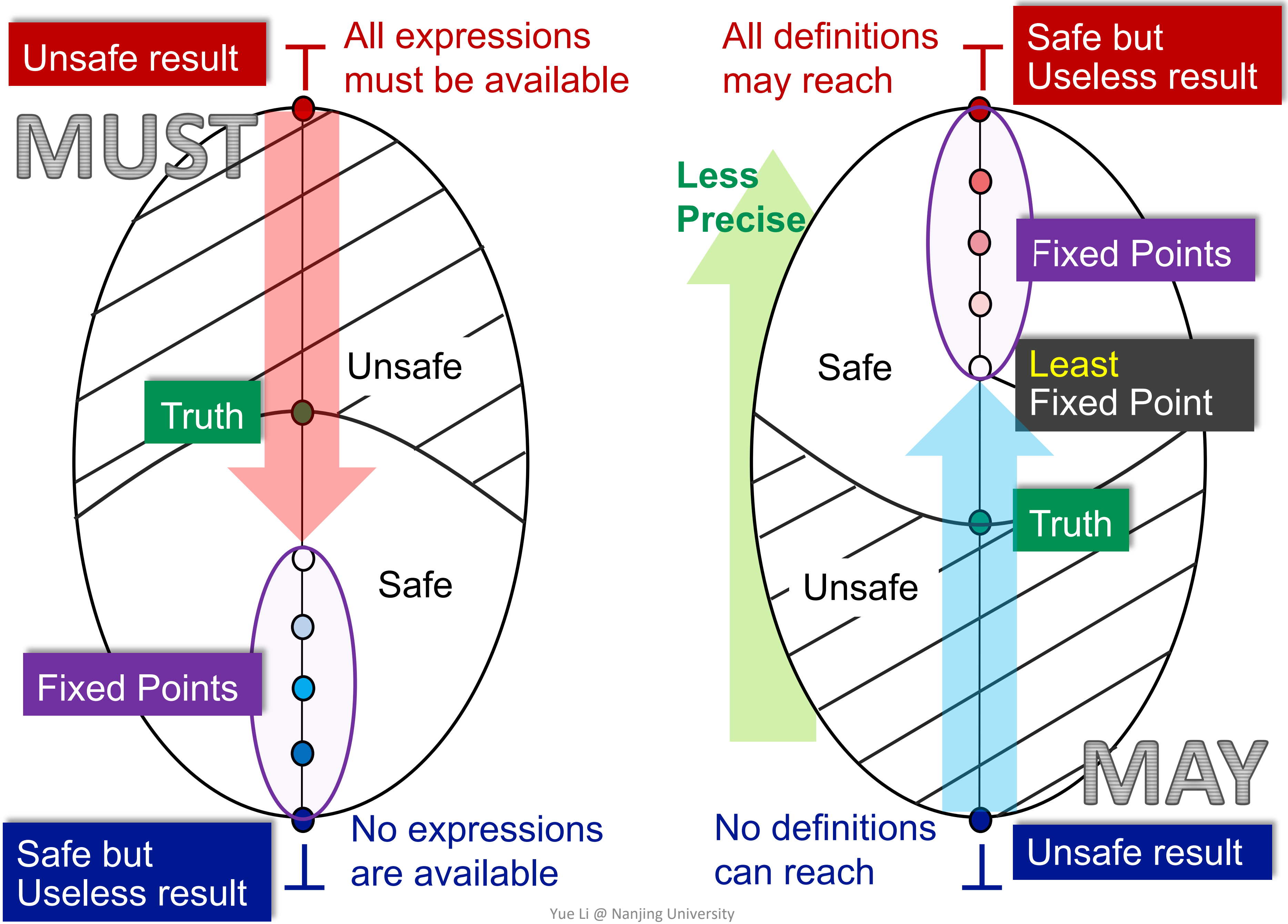

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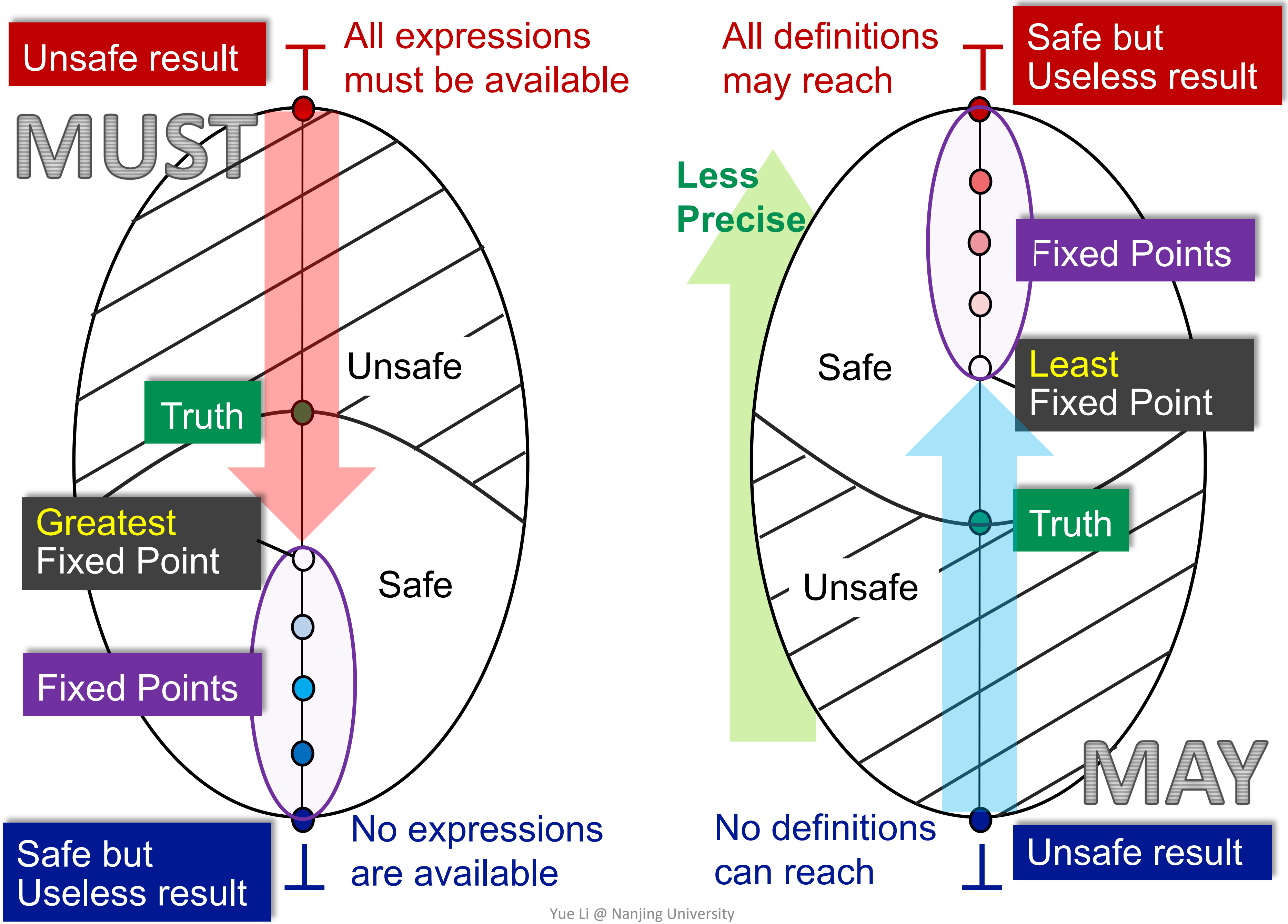
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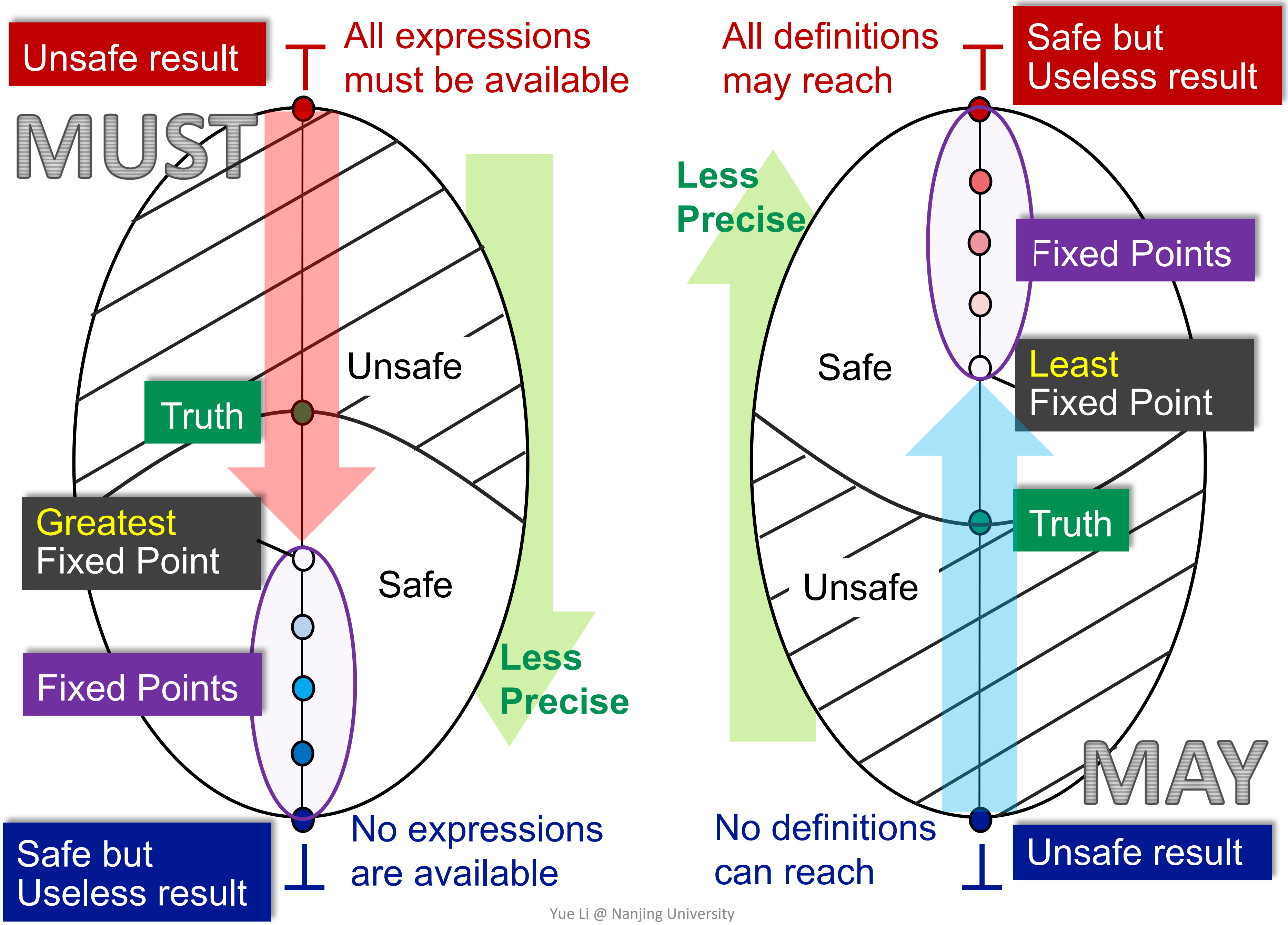
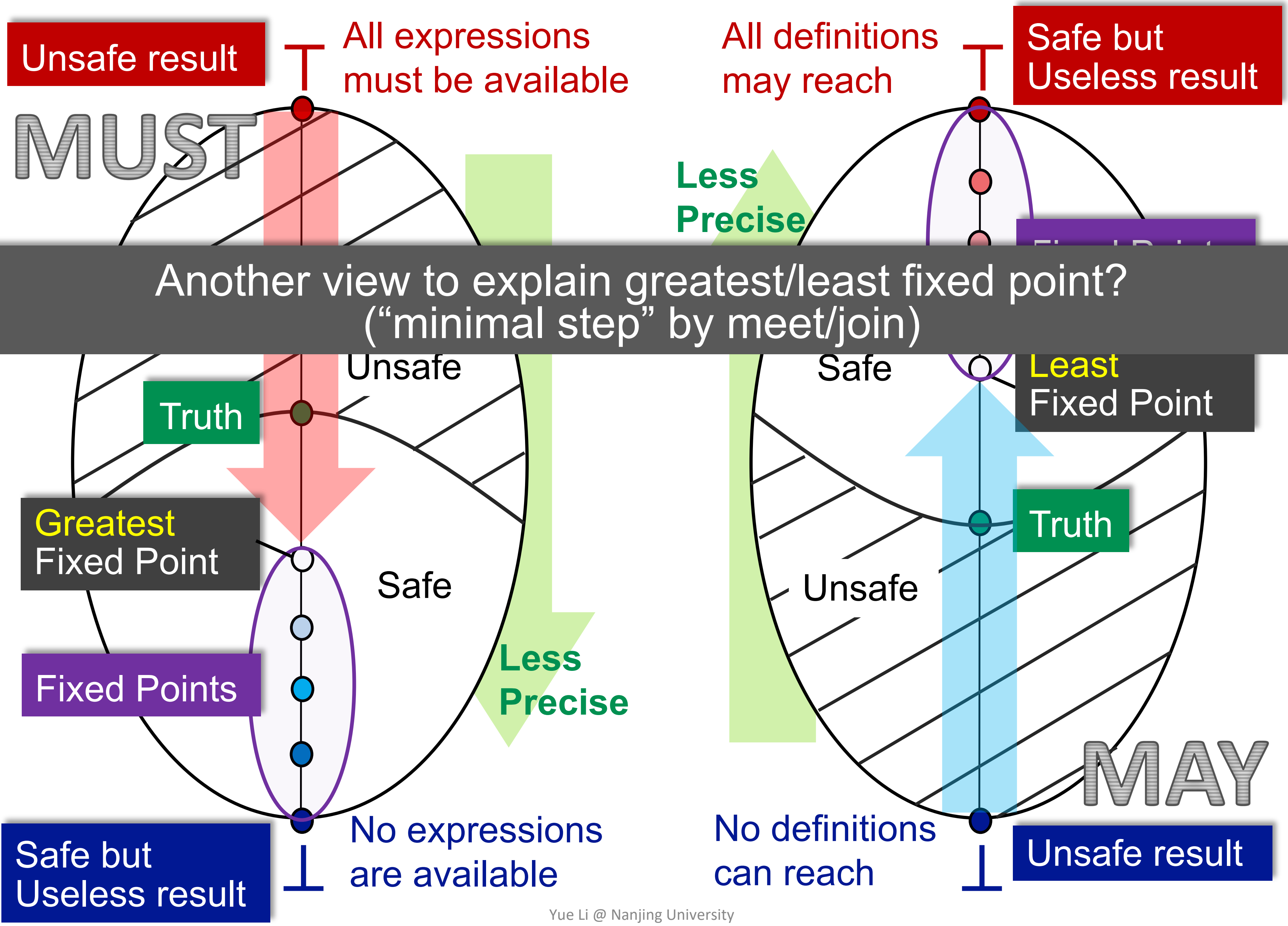
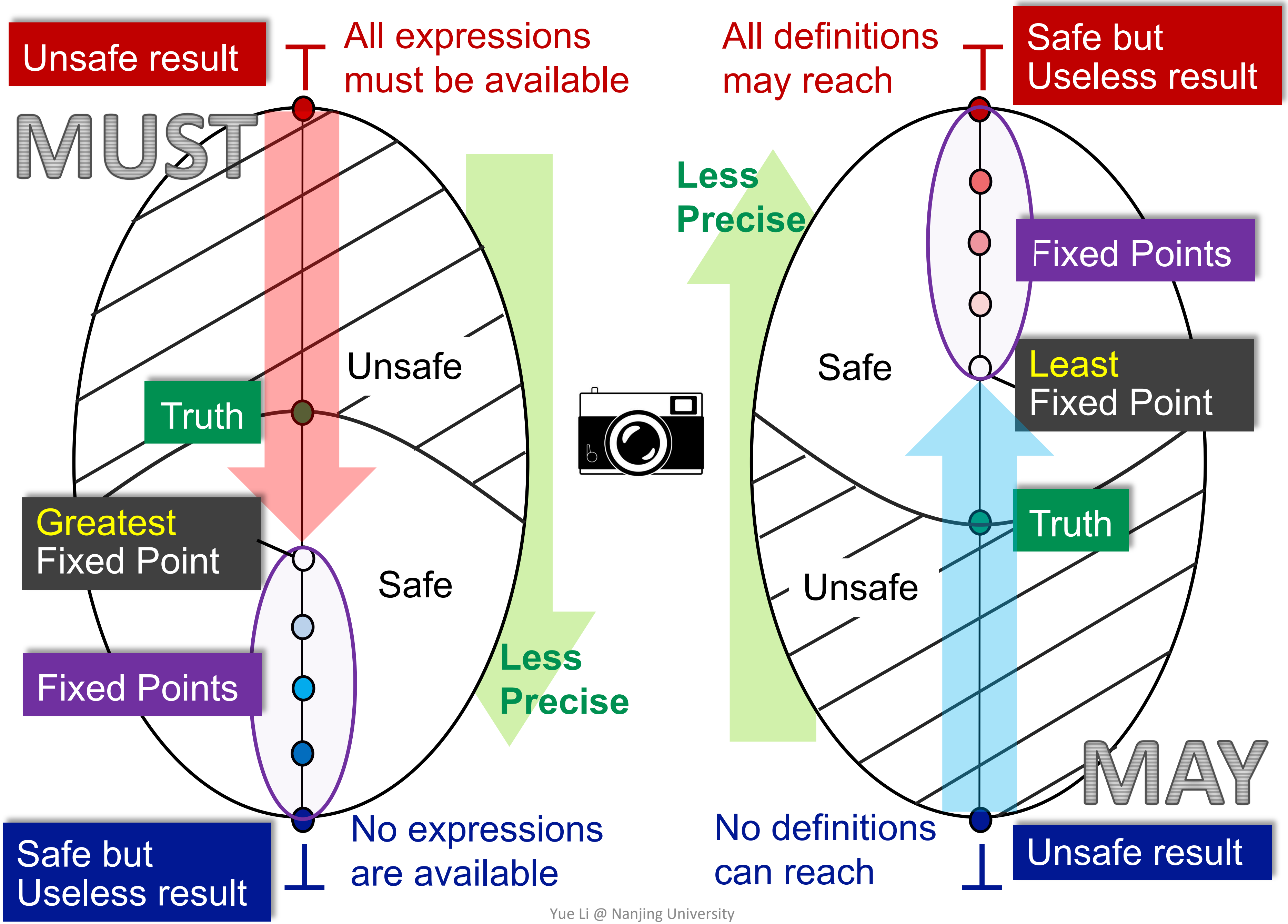


Yue Li @ Nanjing University

Another view to explain greatest/least fixed point?
(“minimal step” by meet/join)





MUST

Unsafe result

⊤

All expressions must be available

Unsafe

Truth

Safe

Greatest Fixed Point

Fixed Points

Less Precise

Safe but Useless result

⊥

No expressions are available

MAY

All definitions may reach

⊤

Safe but Useless result

Less Precise

Fixed Points

Safe

Least Fixed Point

Truth

Unsafe

No definitions can reach

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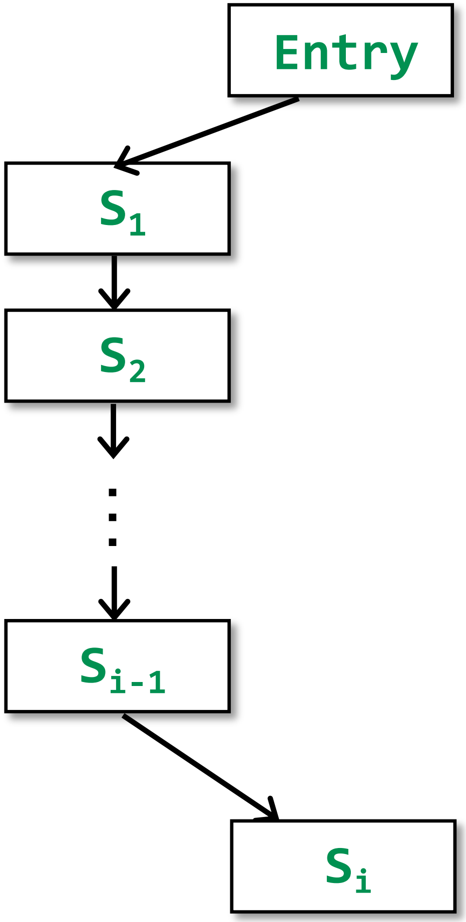
Unsafe result

How Precise Is Our Solution?

- Meet-Over-All-Paths Solution (MOP)

How Precise Is Our Solution?

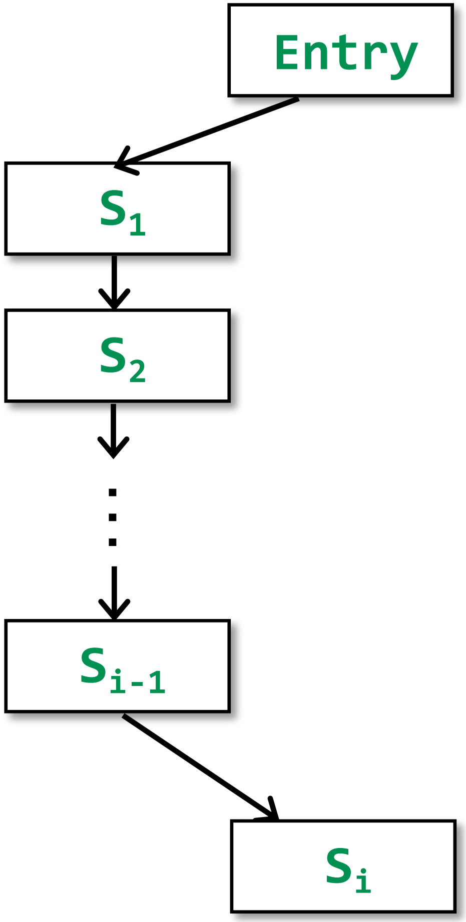
- Meet-Over-All-Paths Solution (MOP)



P = Entry → S_1 → S_2 → … → S_i

How Precise Is Our Solution?

- Meet-Over-All-Paths Solution (MOP)

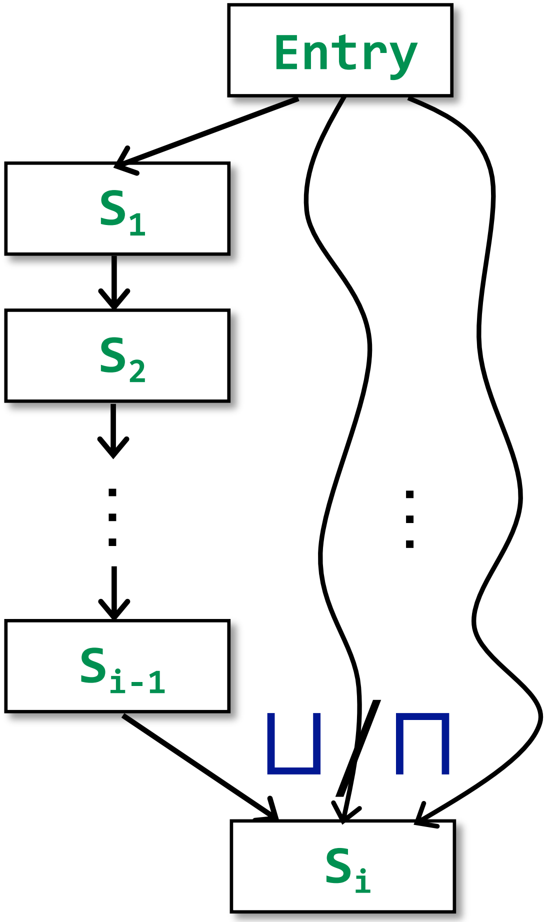


P = Entry → S_1 → S_2 → … → S_i

Transfer function F_P for a path P (from Entry to S_i) is a composition of transfer functions for all statements on that path: f_{S1}, f_{S2}, …, f_{Si-1}

How Precise Is Our Solution?

- Meet-Over-All-Paths Solution (MOP)



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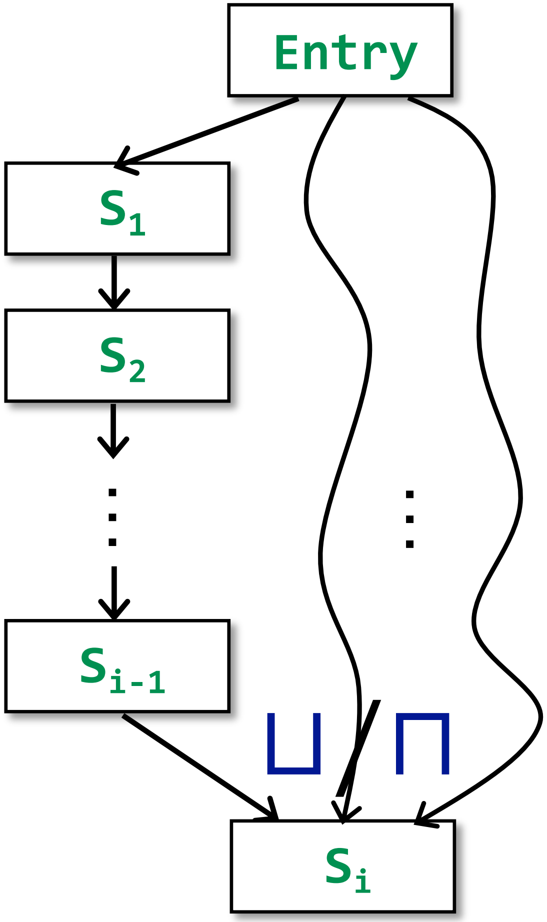
$$\boxed{\mathrm{MOP}[S_i] = \bigsqcup_{\text{A path } P \text{ from } Entry \text{ to } S_i} / \bigsqcap_{\text{A path } P \text{ from } Entry \text{ to } S_i} F_P(\mathrm{OUT}[Entry])}$$

How Precise Is Our Solution?

- Meet-Over-All-Paths Solution (MOP)



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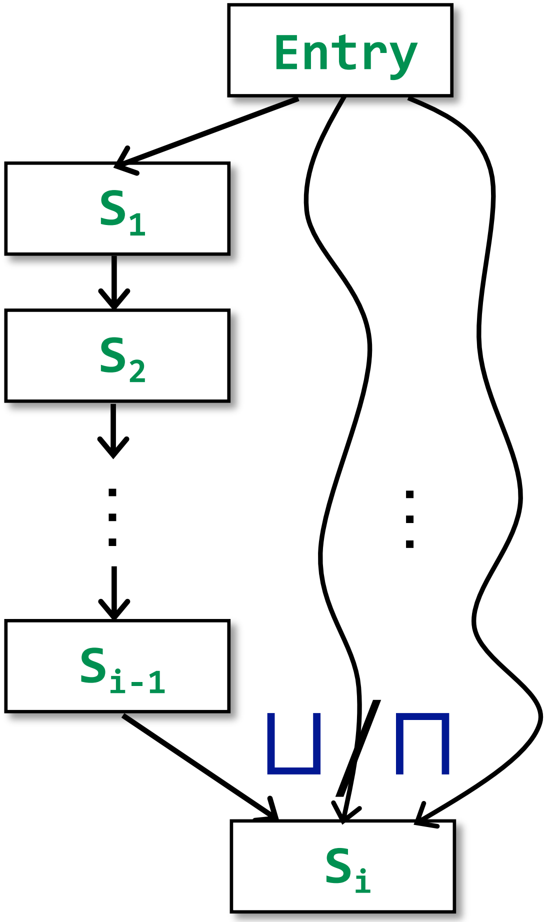
MOP computes the data-flow values at the end of each path and apply join / meet operator to these values to find their lub / glb

How Precise Is Our Solution?

- Meet-Over-All-Paths Solution (MOP)



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Some paths may be not executable → **not fully precise**

How Precise Is Our Solution?

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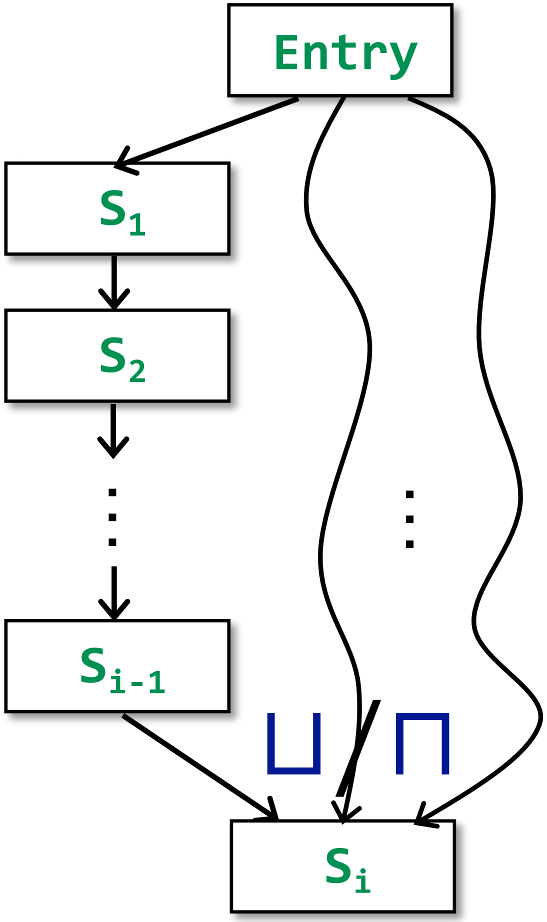

$$P = \texttt{Entry} \rightarrow S_1 \rightarrow S_2 \rightarrow \dots \rightarrow S_i$$

Transfer function F_P for a path P (from Entry to S_i) is a composition of transfer functions for all statements on that path: f_{S1}, f_{S2}, …, f_{Si-1}

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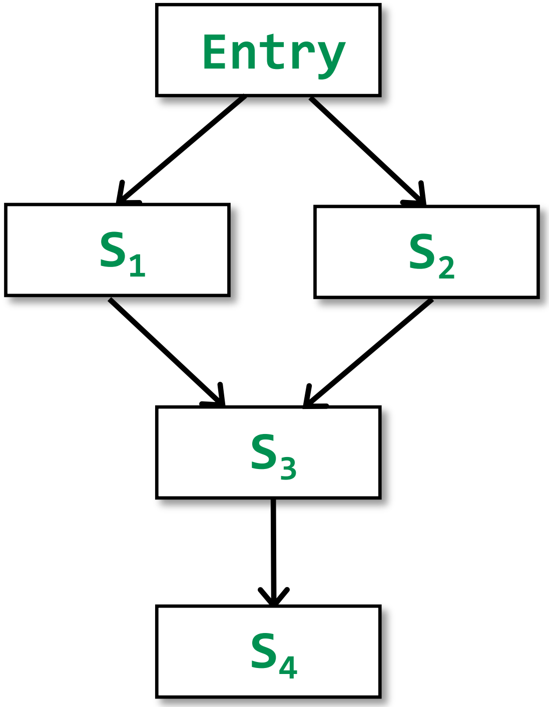
MOP computes the data-flow values at the end of each path and apply join / meet operator to these values to find their lub / glb

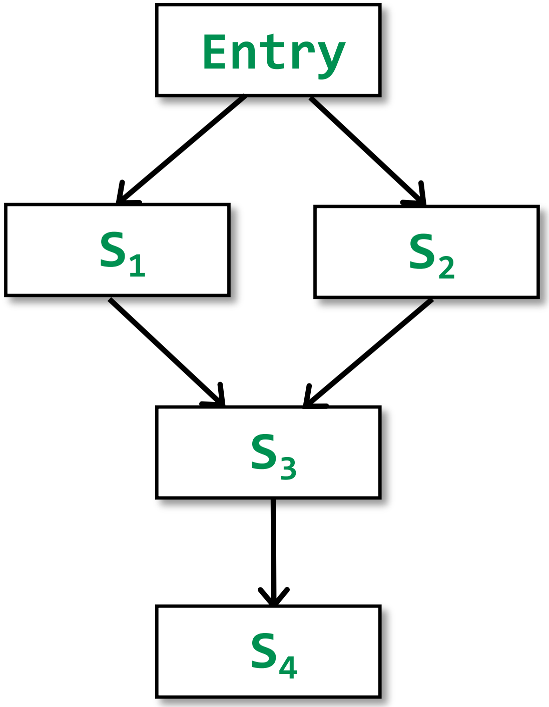
Some paths may be not executable → **not fully precise**

Unbounded, and not enumerable → **impractical**

Ours (Iterative Algorithm) vs. MOP

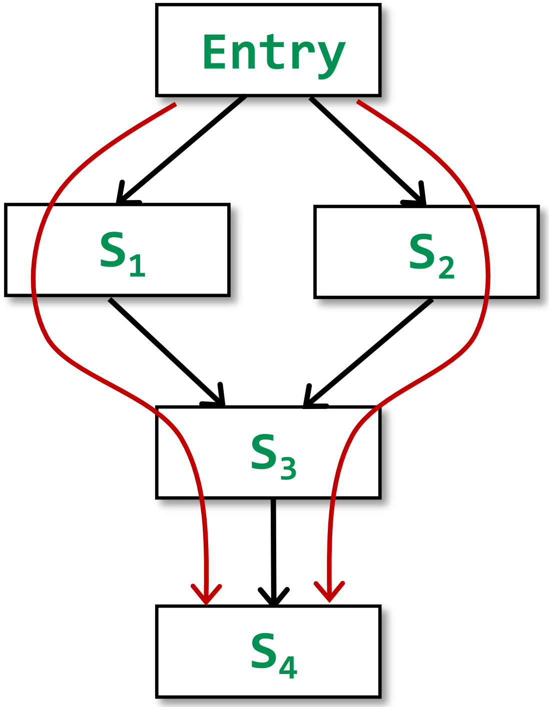


Ours (Iterative Algorithm) vs. MOP



$$\mathrm{IN}[S_4] = f_{S_3}(f_{S_1}(\mathrm{OUT}[\mathtt{Entry}]) \sqcup f_{S_2}(\mathrm{OUT}[\mathtt{Entry}]))$$

Ours (Iterative Algorithm) vs. MOP



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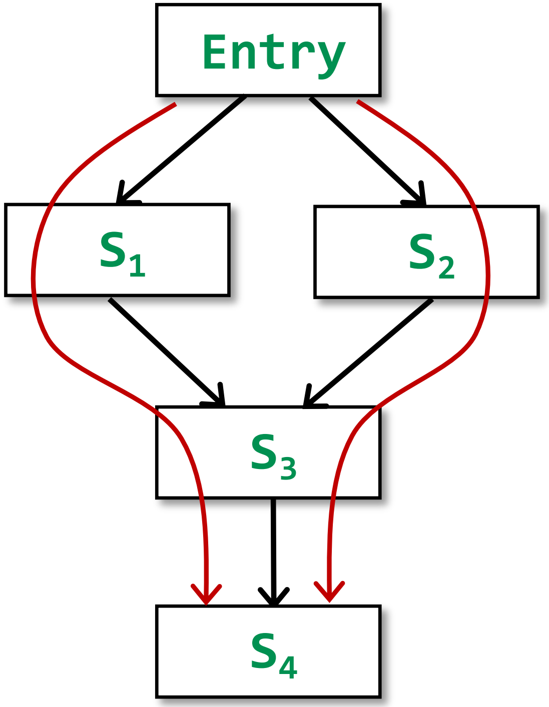

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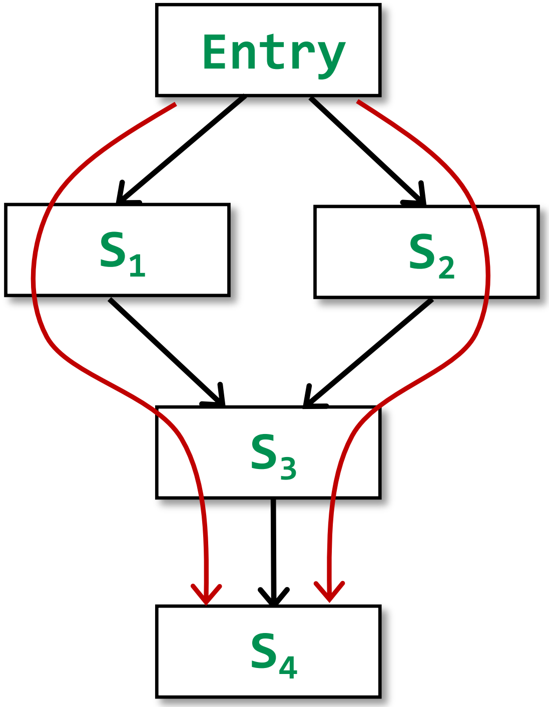
Ours (Iterative Algorithm) vs. MOP



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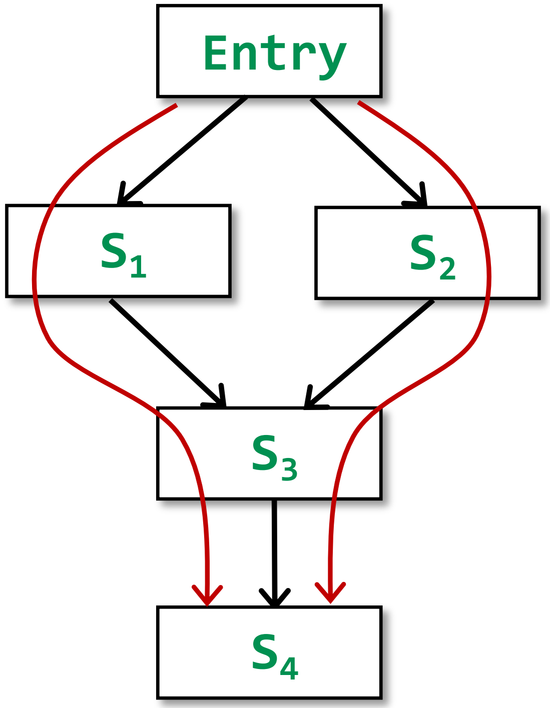

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$F(x \sqcup y)$

M

(Ours is less precise tha

Bit-vector or Gen/Kill problems (set union /intersection for join/meet) are distributive

When **F is distributive**, i.e.,

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MOP = Ours

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Bit-vector or Gen/Kill prob
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But some analyses are not distributive

Constant Propagation

Given a variable x at program point p, determine whether x is guaranteed to hold a constant value at p.

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A data flow analysis framework (D, L, F) consists of:

- **D**: a direction of data flow: forwards or backwards
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Constant Propagation – Lattice

- Domain of the values V

- Meet Operator ⊓

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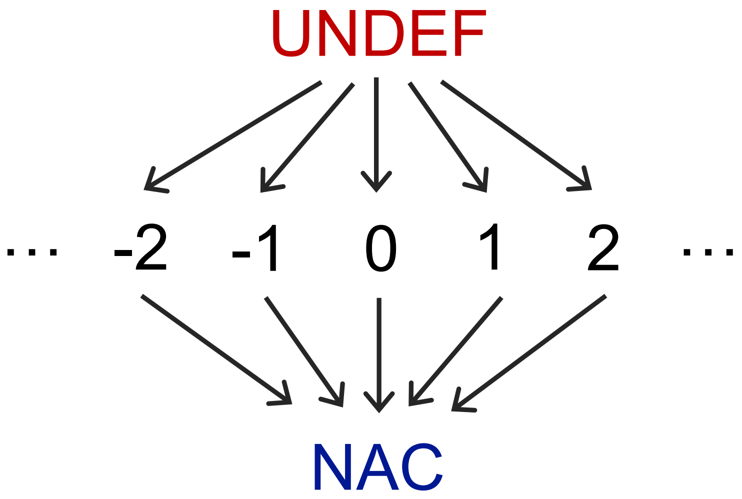
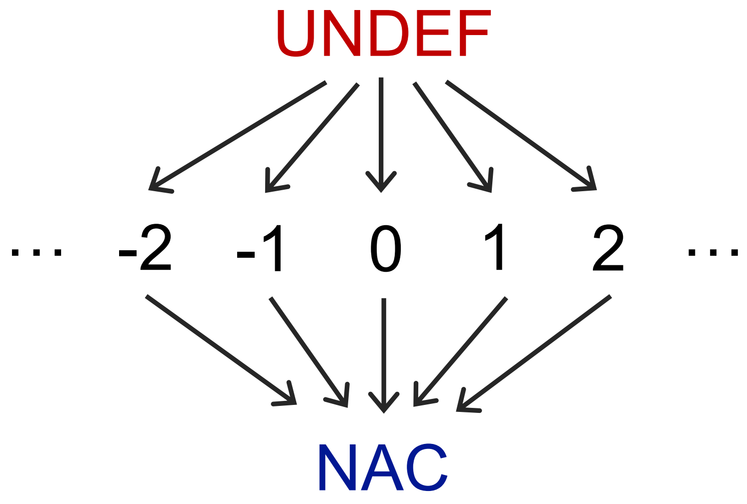


Constant Propagation – Lattice

- Domain of the values V

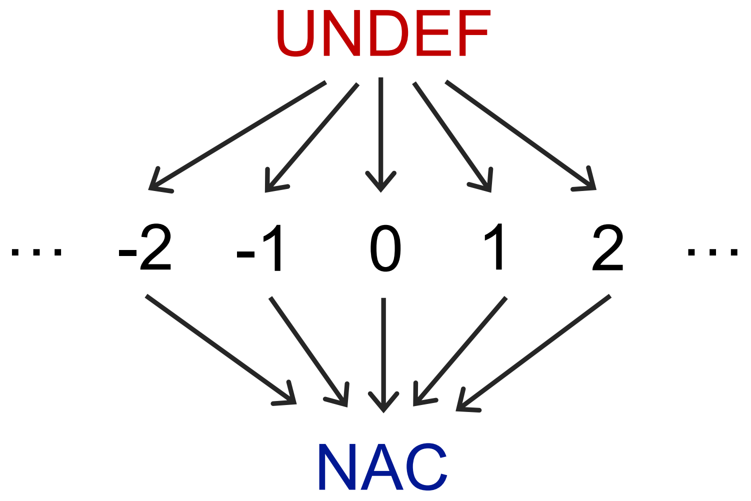


- Meet Operator ⊓

NAC ⊓ v = NAC

Constant Propagation – Lattice

- Domain of the values V



UNDEF

… -2 -1 0 1 2 …

NAC

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Constant Propagation – Lattice

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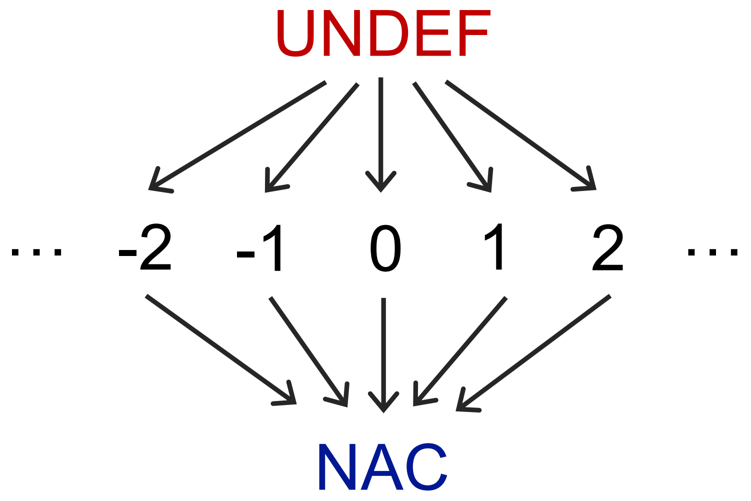

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Uninitialized variables are not the focus in our constant propagation analysis

Constant Propagation – Lattice

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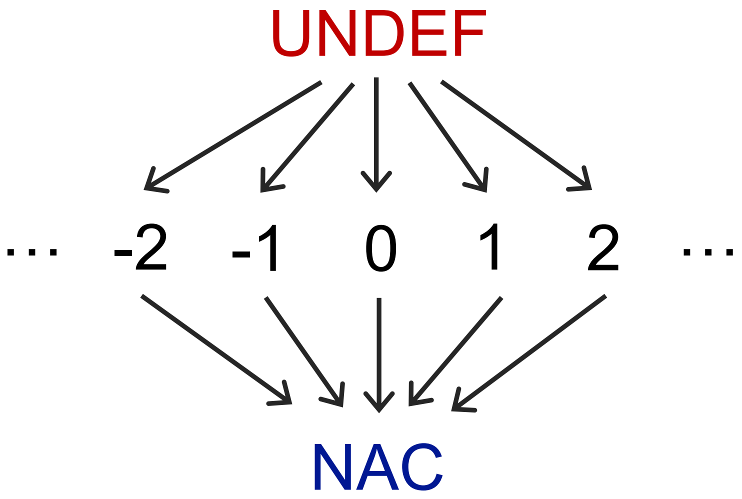

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$\text{NAC} \sqcap v = \text{NAC}$

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$c \sqcap v = ?$

Constant Propagation – Lattice

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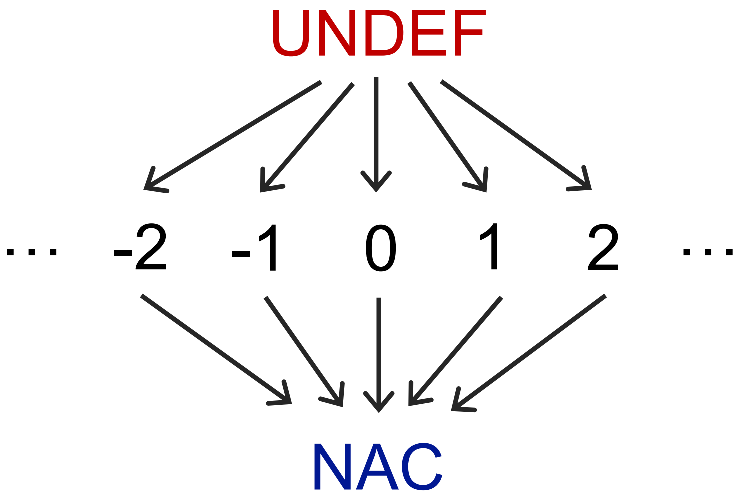

- Meet Operator ⊓

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UNDEF ⊓ v = v

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c ⊓ v = ?

- c ⊓ c = c
- c_1 ⊓ c_2 = NAC

Constant Propagation – Lattice

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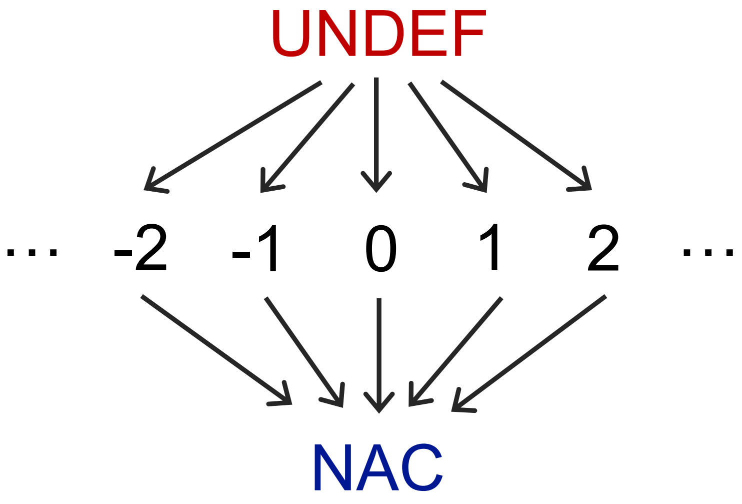

- Meet Operator $\sqcap$

NAC $\sqcap$ v = NAC

UNDEF $\sqcap$ v = v

Uninitialized variables are not the focus in our constant propagation analysis

$c \sqcap v$ = ?

- $c \sqcap c$ = c
- $c_1 \sqcap c_2$ = NAC

At each path confluence PC, we should apply “meet” for all variables in the incoming data-flow values at that PC

Constant Propagation – Transfer Function

Given a statement s: x = …, we define its transfer function F as

$$\boxed{F: \mathrm{OUT}[s] = \mathrm{gen} \cup (\mathrm{IN}[s] - \{(x, \_)\})}$$

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- s: x = y;

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• s: x = y;	gen = {(x, val(y))}
• s: x = y *op* z;	gen = {(x, f(y,z))}

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$$f(y,z) = \begin{cases} val(y)\ op\ val(z) & \text{// if val(y) and val(z) are constants} \\ NAC & \text{// if val(y) or val(z) is NAC} \\ UNDEF & \text{// otherwise} \end{cases}$$

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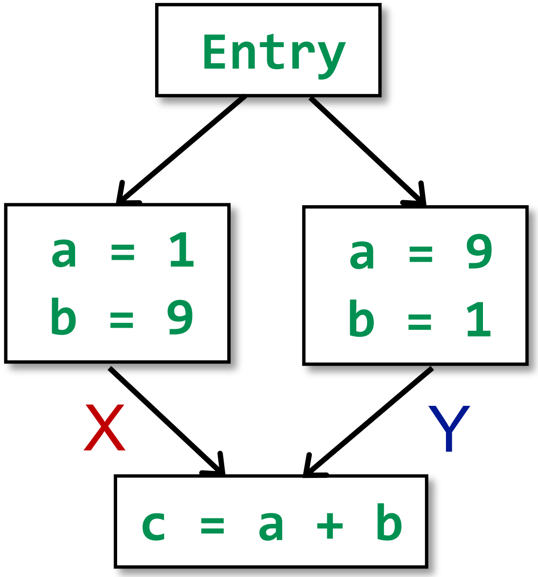
(if s is not an assignment statement, F is the identity function)

Constant Propagation – Nondistributivity



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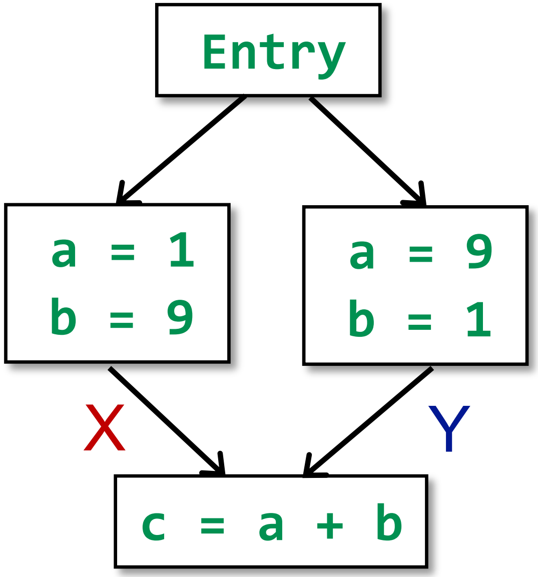

$F(X \sqcap Y) =$

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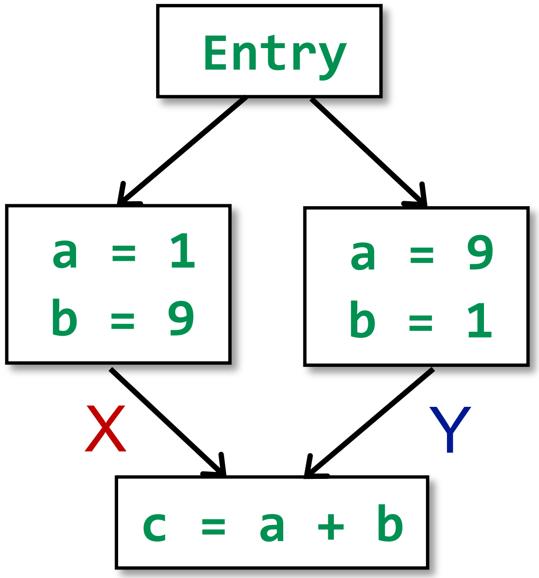
Constant Propagation – Nondistributivity



$F(X \sqcap Y) = \{(a, NAC), (b, NAC), (c, NAC)\}$

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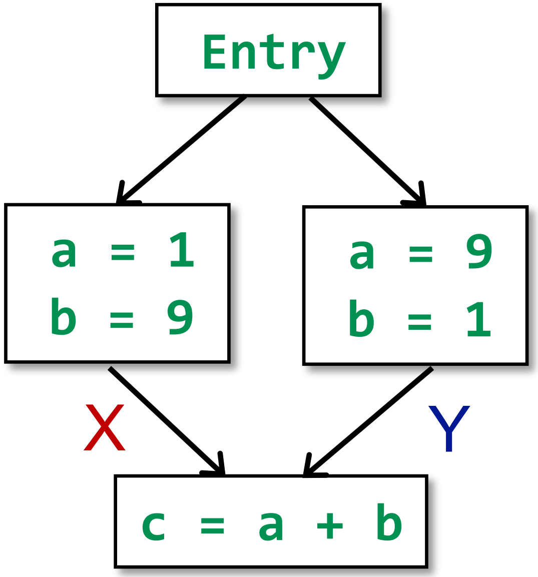
Constant Propagation – Nondistributivity



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$F(X) \sqcap F(Y) = \{(a, NAC), (b, NAC), (c, 10)\}$

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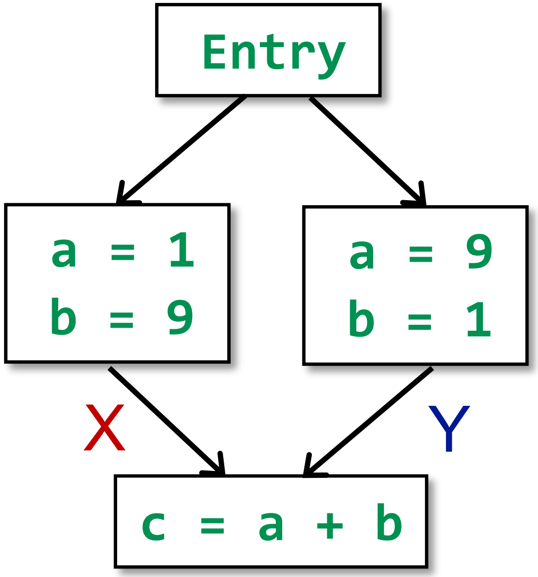

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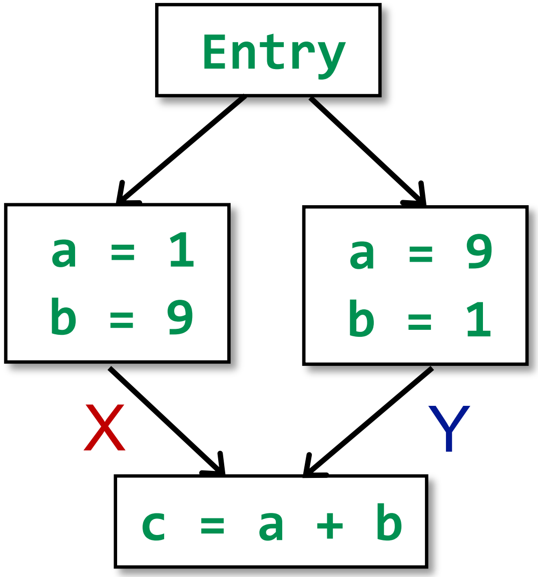
Constant Propagation – Nondistributivity



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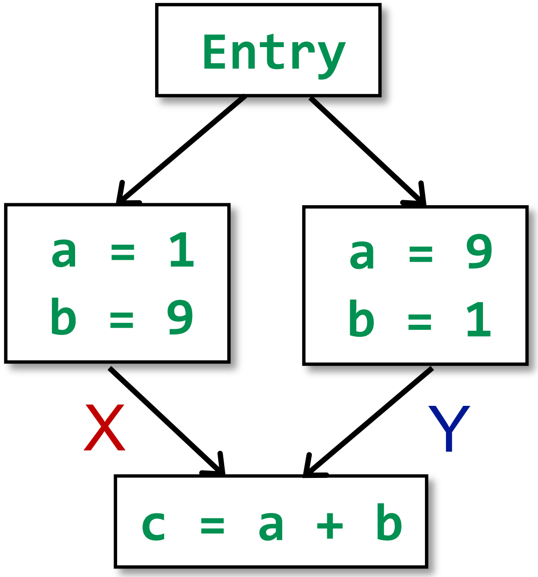

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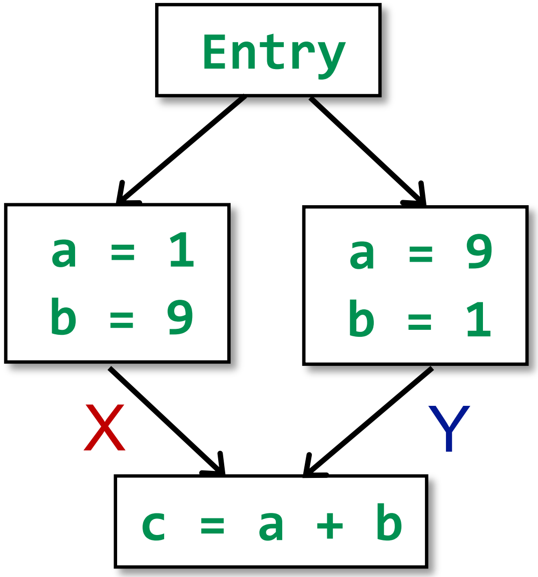
Show our constant propagation analysis is monotonic

Constant Propagation – Nondistributivity



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Show our constant propagation analysis is monotonic

Assignment One: Constant Propagation

Worklist Algorithm,

an optimization of Iterative Algorithm

Review Iterative Algorithm for May & Forward Analysis

INPUT: CFG ($kill_B$ and gen_B computed for each basic block B)

OUTPUT: IN[B] and OUT[B] for each basic block B

METHOD:

```
OUT[entry] = ∅;
for (each basic block B\entry)
    OUT[B] = ∅;
while (changes to any OUT occur)
    for (each basic block B\entry) {
        IN[B] = $\bigsqcup_{P \text{ a predecessor of } B}$ OUT[P];
        OUT[B] = $gen_B$ U (IN[B] - $kill_B$);
    }
```

Worklist Algorithm

Forward Analysis

```
OUT[entry] = $\emptyset$;
for (each basic block $B \setminus entry$)
    OUT[B] = $\emptyset$;
Worklist ← all basic blocks
while (Worklist is not empty)
    Pick a basic block B from Worklist
    old_OUT = OUT[B]
    IN[B] = $\bigsqcup_{P\ a\ predecessor\ of\ B}$ OUT[P];
    OUT[B] = $gen_B \cup (IN[B] - kill_B)$;
    if (old_OUT ≠ OUT[B])
        Add all successors of B to Worklist
```

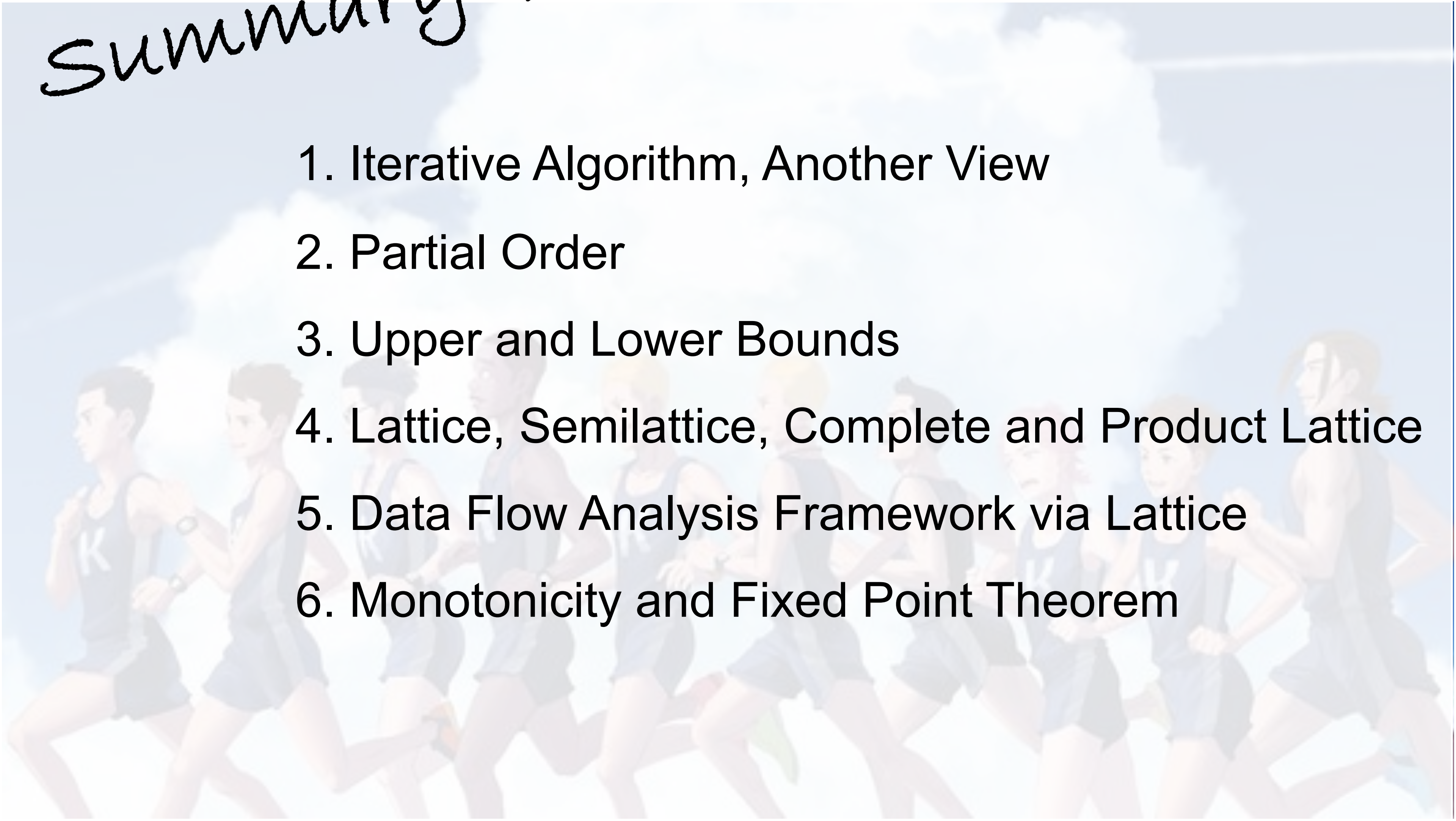
Worklist Algorithm

Forward Analysis

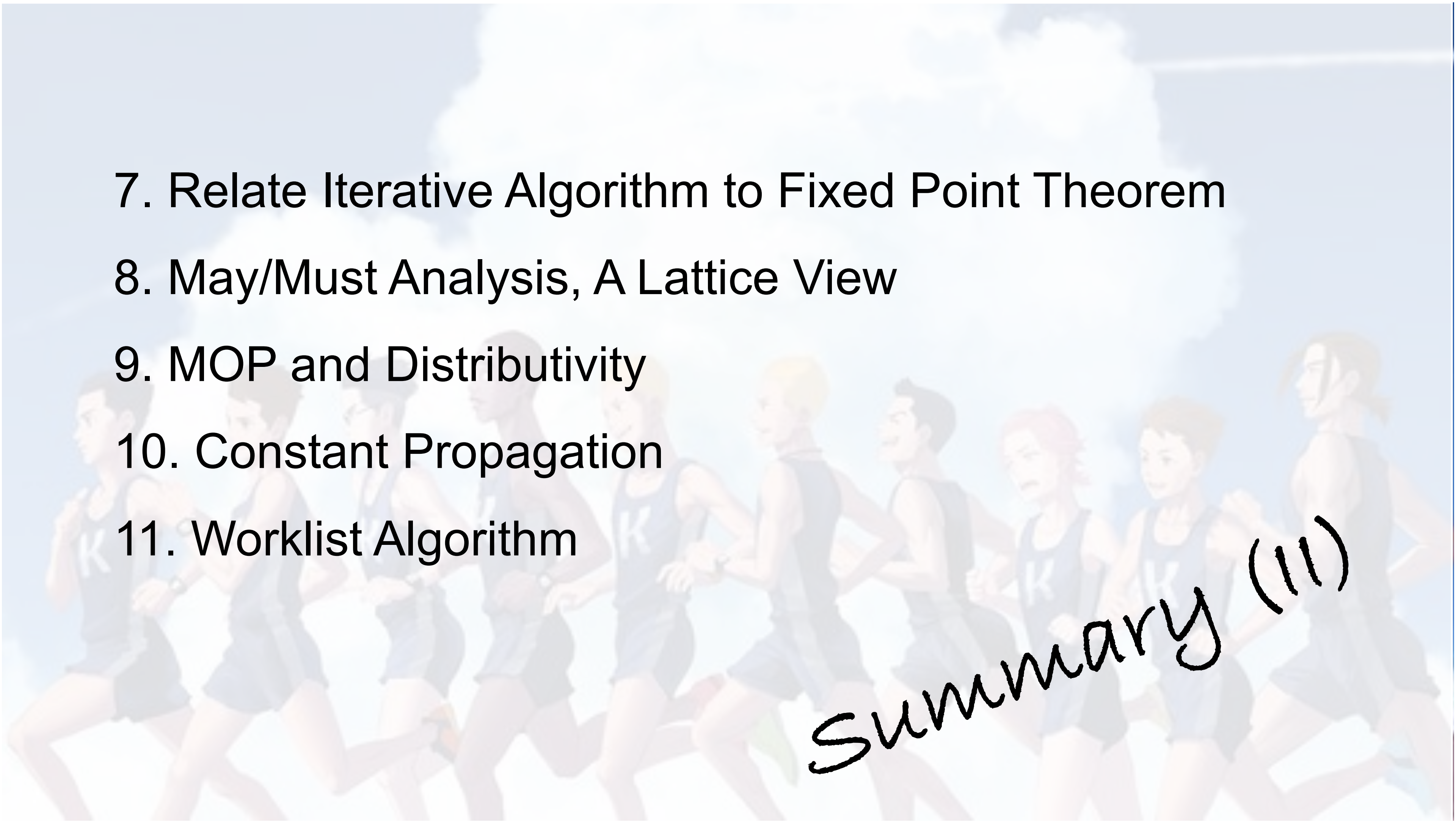
```
$OUT[entry] = \emptyset$;
for (each basic block $B \setminus entry$)
    $OUT[B] = \emptyset$;
Worklist ← all basic blocks
while (Worklist is not empty)
    Pick a basic block B from Worklist
    old_OUT = OUT[B]
    $IN[B] = \sqcup_{P \text{ a predecessor of } B} OUT[P]$;
    $OUT[B] = gen_B \cup (IN[B] - kill_B)$;
    if (old_OUT ≠ OUT[B])
        Add all successors of B to Worklist
```

OUT will not change if IN does not change

Summary (I)



1. Iterative Algorithm, Another View
2. Partial Order
3. Upper and Lower Bounds
4. Lattice, Semilattice, Complete and Product Lattice
5. Data Flow Analysis Framework via Lattice
6. Monotonicity and Fixed Point Theorem



7. Relate Iterative Algorithm to Fixed Point Theorem
8. May/Must Analysis, A Lattice View
9. MOP and Distributivity
10. Constant Propagation
11. Worklist Algorithm

Summary (II)

The X You Need To Understand in This Lecture

- Understand the functional view of iterative algorithm
- The definitions of lattice and complete lattice
- Understand the fixed-point theorem
- How to summarize may and must analyses in lattices
- The relation between MOP and the solution produced by the iterative algorithm
- Constant propagation analysis
- Worklist algorithm