Static Program Analysis

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Static Program Analysis

Data Flow Analysis — Foundations

Nanjing University

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Contents (1)

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Let us first recall the iterative algorithm for data flow analysis

This general iterative algorithm produces a solution to data flow analysis

Iterative Algorithm for May & Forward Analysis

INPUT: CFG ($kill_B$ and gen_B computed for each basic block B)

OUTPUT: IN[B] and OUT[B] for each basic block B

METHOD:

```
OUT[entry] = ∅;
for (each basic block B\entry)
    OUT[B] = ∅;
while (changes to any OUT occur)
    for (each basic block B\entry) {
        IN[B] = ∪_{P a predecessor of B} OUT[P];
        OUT[B] = gen_B ∪ (IN[B] - kill_B);
    }
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- Given a CFG (program) with k nodes, the iterative algorithm updates OUT[n] for every node n in each iteration.

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- Assume the domain of the values in data flow analysis is V, then we can define a k-tuple

$$(\mathrm{OUT}[n_1], \mathrm{OUT}[n_2], \ldots, \mathrm{OUT}[n_k])$$

as an element of set $(V_1 \times V_2 \ldots \times V_k)$ denoted as V^k , to hold the values of the analysis after each iteration.

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- Each iteration can be considered as taking an action to map an element of V^k to a new element of V^k, through applying the transfer functions and control-flow handing, abstracted as a function $F: V^k \rightarrow V^k$
- Then the algorithm outputs a series of k-tuples iteratively until a k-tuple is the same as the last one in two consecutive iterations

OUT[*entry*] = ∅;
for (each basic block *B**entry*)
    OUT[*B*] = ∅;
while (changes to any OUT occur)
    for (each basic block *B**entry*) {
        IN[*B*] = $\bigcup_{P \text{ a predecessor of } B}$ OUT[*P*];
        OUT[*B*] = gen_B U (IN[*B*] - $kill_B$);
    }

Given a CFG (program) with k nodes, the iterative algorithm updates OUT[n] for every node n in each iteration.

$$init \rightarrow (\bot, \ \bot, \ \ldots, \ \bot)$$

```
OUT[entry] = Ø;
for (each basic block B\entry)
    OUT[B] = Ø;
while (changes to any OUT occur)
    for (each basic block B\entry) {
        IN[B] = U_{P a predecessor of B} OUT[P];
        OUT[B] = gen_B U (IN[B] - kill_B);
    }
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Given a CFG (program) with k nodes, the iterative algorithm updates OUT[n] for every node n in each iteration.

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Given a CFG (program) with k nodes, the iterative algorithm updates OUT[n] for every node n in each iteration.

$$\textit{init} \rightarrow (\bot,\ \bot,\ \ldots,\ \bot)$$

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$$\textit{iter } 2 \rightarrow (v_1^2, v_2^2, \ldots, v_k^2)$$

$$\vdots$$

$$\textit{iter } i \rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$$

OUT[*entry*] = $\emptyset$;
for (each basic block *B**entry*)
    OUT[*B*] = $\emptyset$;
while (changes to any OUT occur)
    for (each basic block *B**entry*) {
        IN[*B*] = $\bigcup_{P \text{ a predecessor of } B}$ OUT[*P*];
        OUT[*B*] = $gen_B \cup (\text{IN}[B] - kill_B)$;
    }

Given a CFG (program) with k nodes, the iterative algorithm updates OUT[n] for every node n in each iteration.

$$
\begin{array}{ll}
init & \rightarrow (\bot, \ \bot, \ \ldots, \ \bot) \\
iter\ 1 & \rightarrow (v_1^1, v_2^1, \ldots, v_k^1) \\
iter\ 2 & \rightarrow (v_1^2, v_2^2, \ldots, v_k^2) \\
 & \quad \vdots \\
iter\ i & \rightarrow (v_1^i, v_2^i, \ldots, v_k^i) \\
iter\ i+1 & \rightarrow (v_1^i, v_2^i, \ldots, v_k^i)
\end{array}
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Given a CFG (program) with k nodes, the iterative algorithm updates OUT[n] for every node n in each iteration.

$$\begin{aligned}
&\textit{init} \rightarrow (\bot, \bot, \ldots, \bot) = X_0 \\
&\textit{iter 1} \rightarrow (v_1^1, v_2^1, \ldots, v_k^1) = X_1 \\
&\textit{iter 2} \rightarrow (v_1^2, v_2^2, \ldots, v_k^2) = X_2 \\
&\qquad \vdots \\
&\textit{iter i} \rightarrow (v_1^i, v_2^i, \ldots, v_k^i) = X_i \\
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Each iteration takes an action $F: V^k \rightarrow V^k$

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\textit{iter 2} & \rightarrow (v_1^2, v_2^2, \ldots, v_k^2) & = X_2 = F(X_1) \\
 & \vdots & \\
\textit{iter i} & \rightarrow (v_1^i, v_2^i, \ldots, v_k^i) & = X_i = F(X_{i-1}) \\
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The iterative algorithm reaches
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To answer these questions, let us learn some math first

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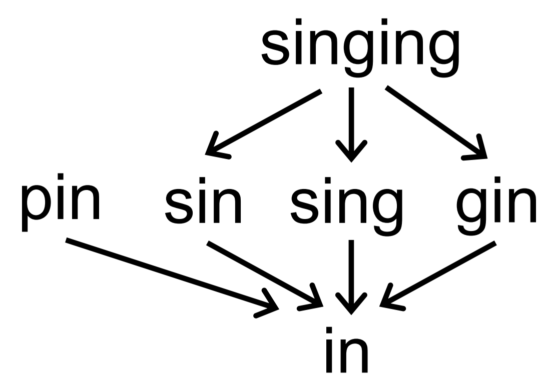
Example 3. Is $(S, \sqsubseteq)$ a poset where S is a set of English words and $\sqsubseteq$ represents the *substring* relation, i.e., s1 $\sqsubseteq$ s2 means s1 is a substring of s2?



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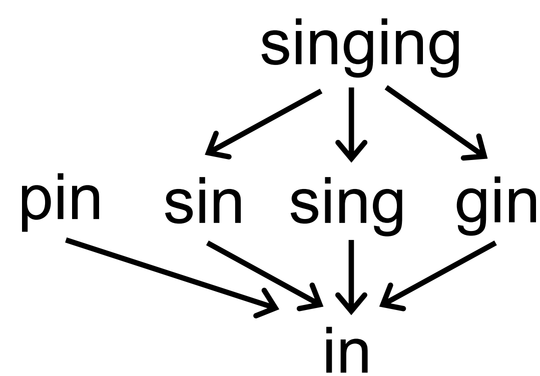

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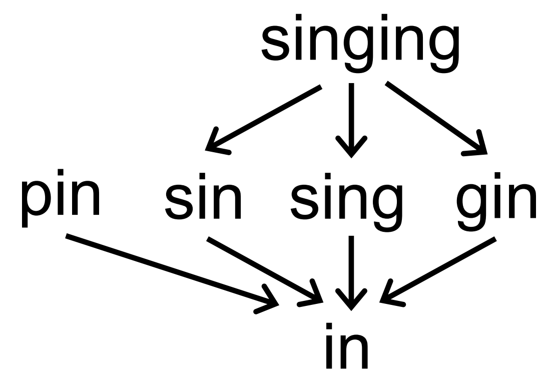

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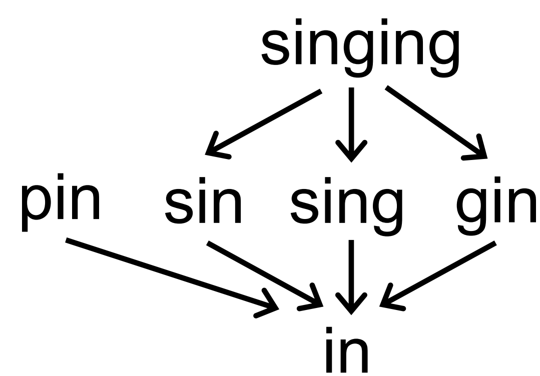

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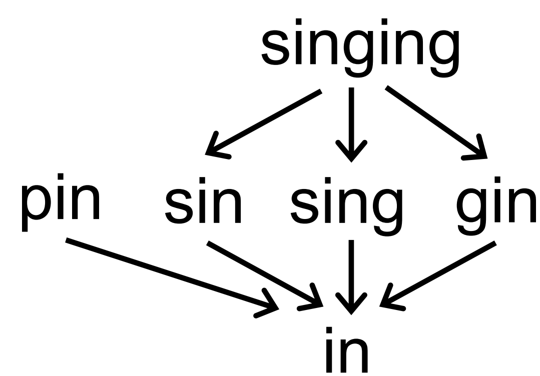

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partial means for a pair of set elements in P, they could be incomparable; in other words, not necessary that every pair of set elements must satisfy the ordering $\sqsubseteq$

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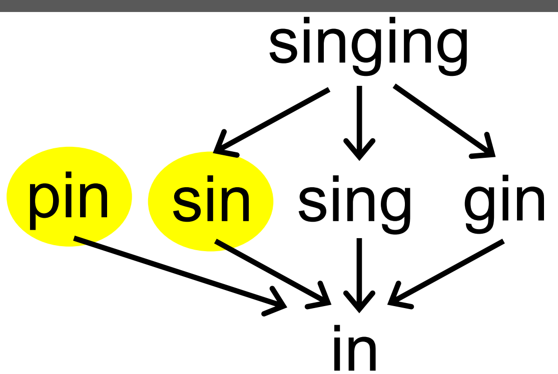

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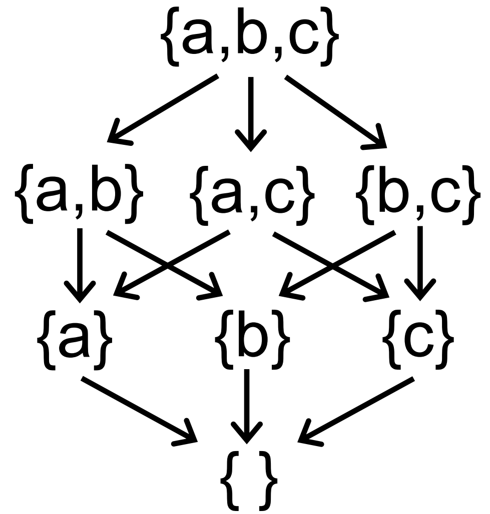

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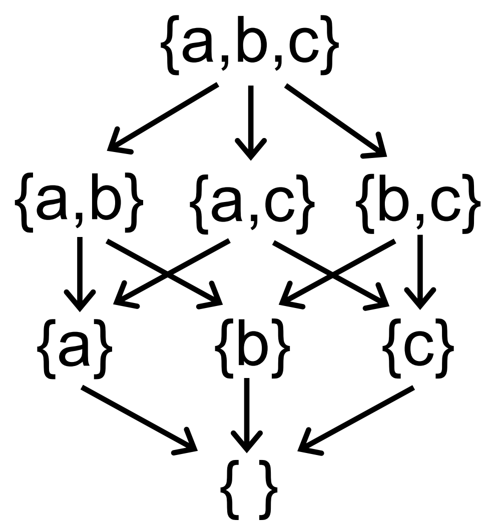

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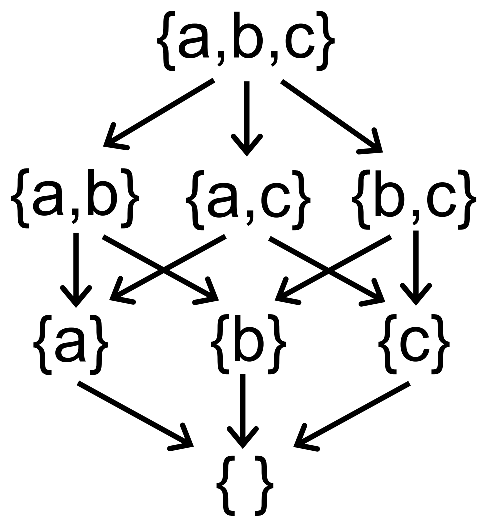

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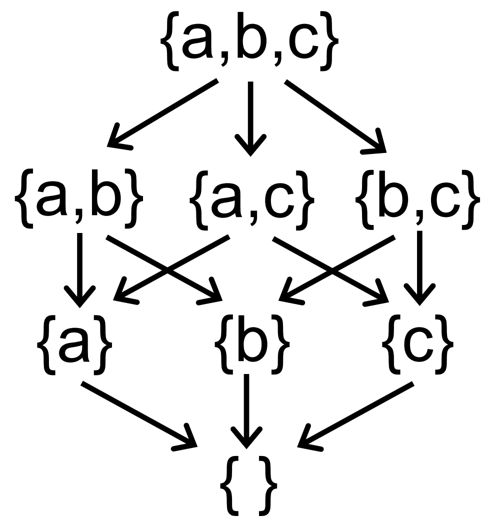

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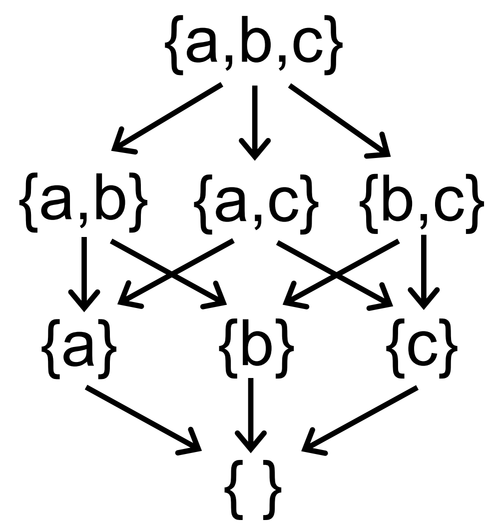

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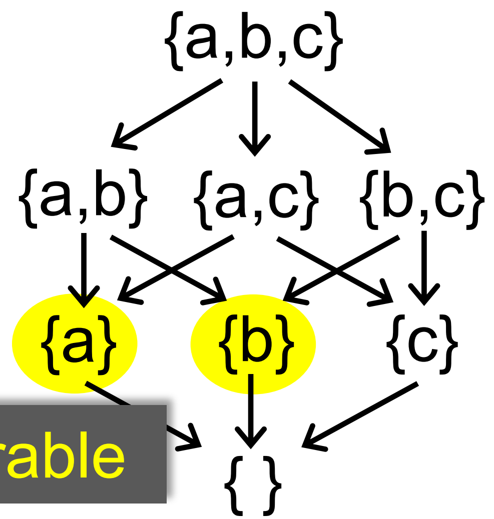

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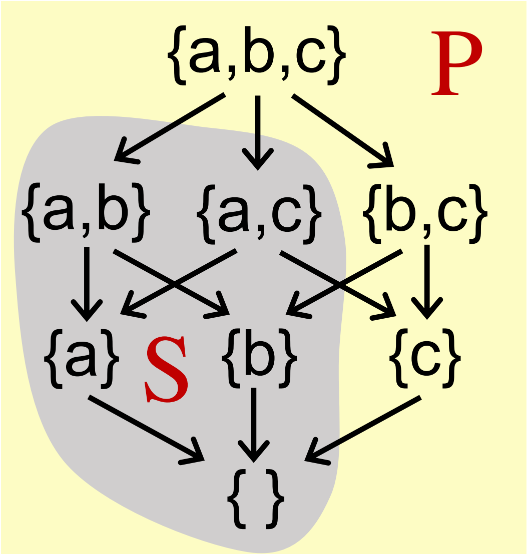

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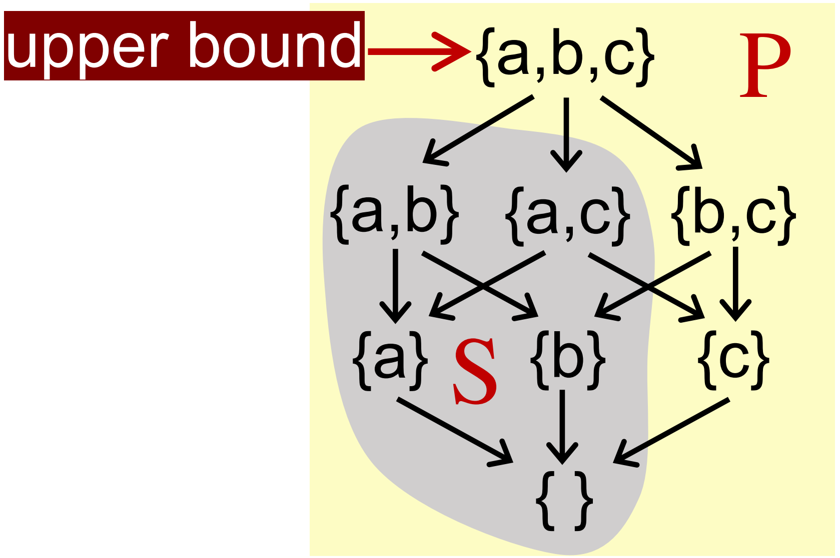

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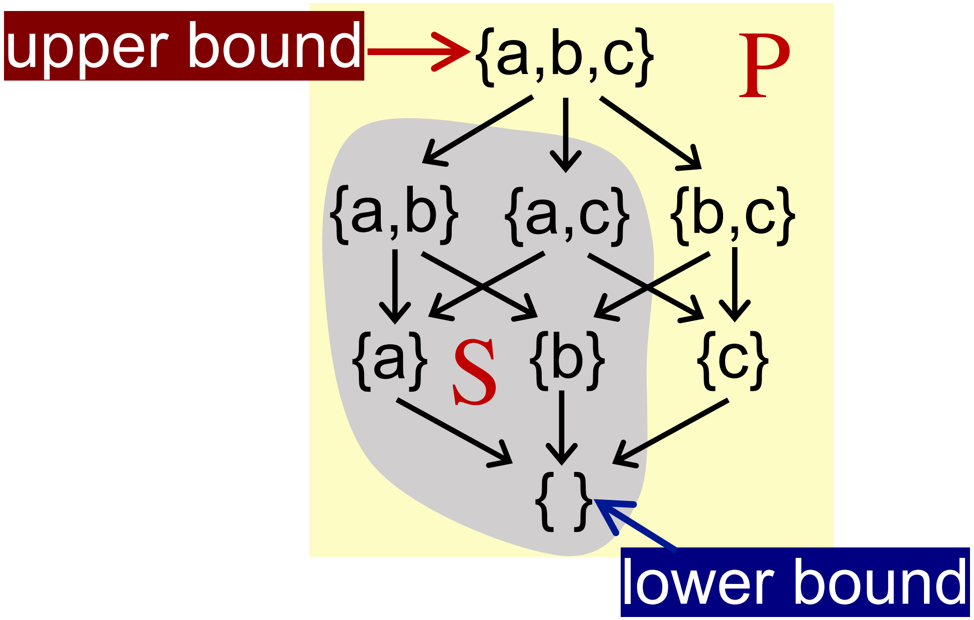

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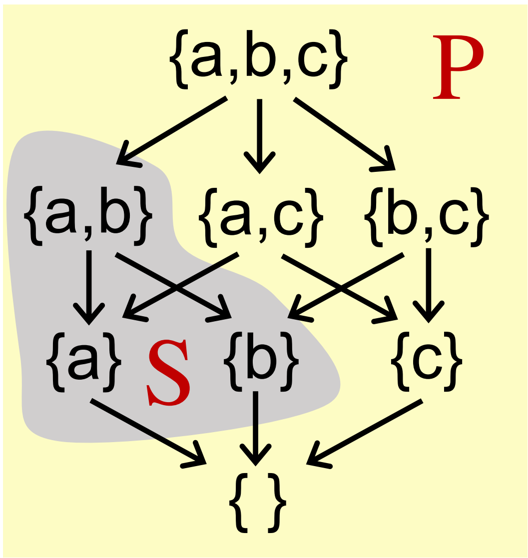

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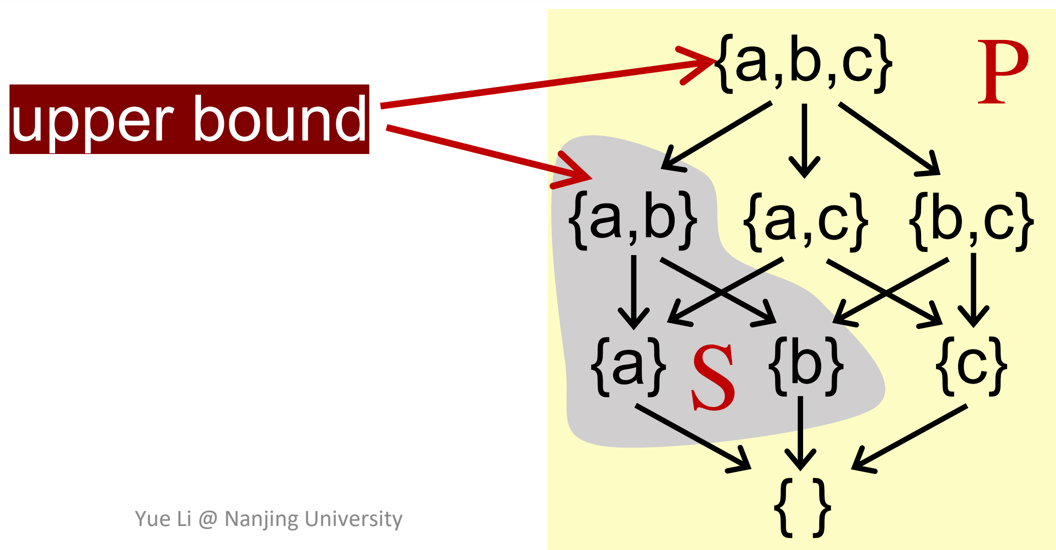

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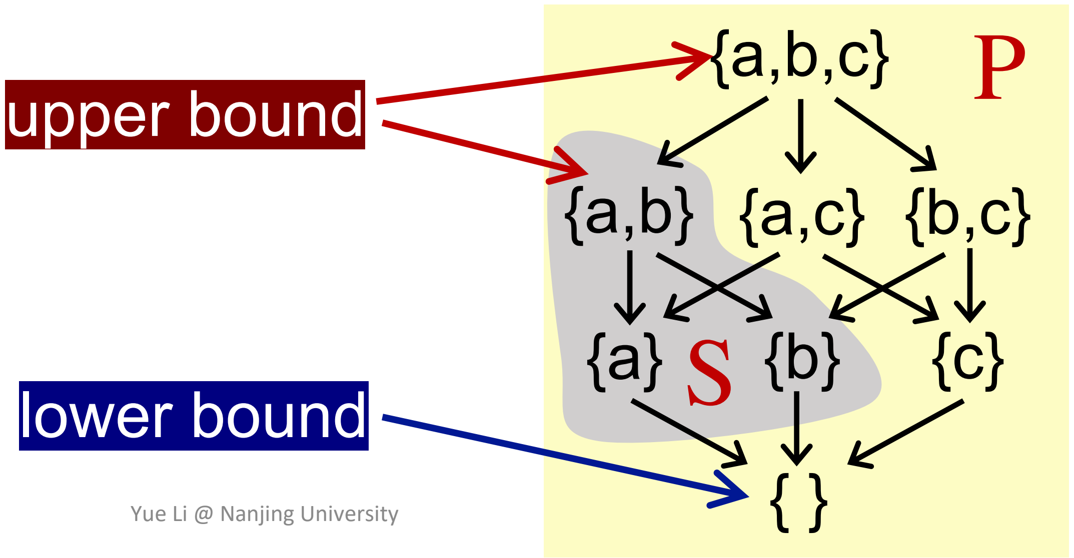

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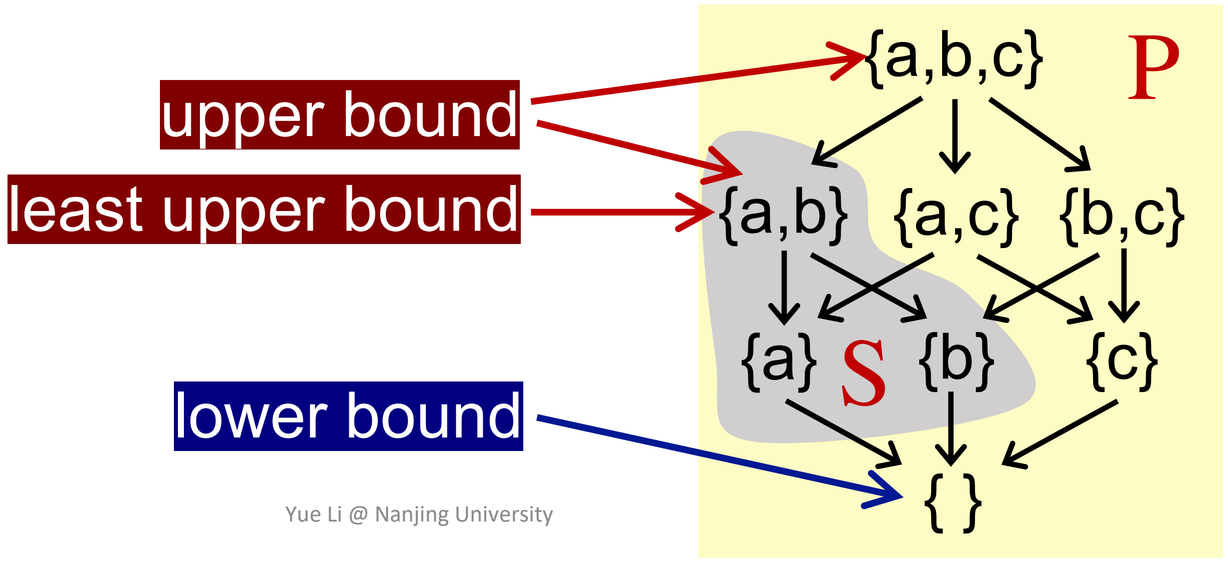

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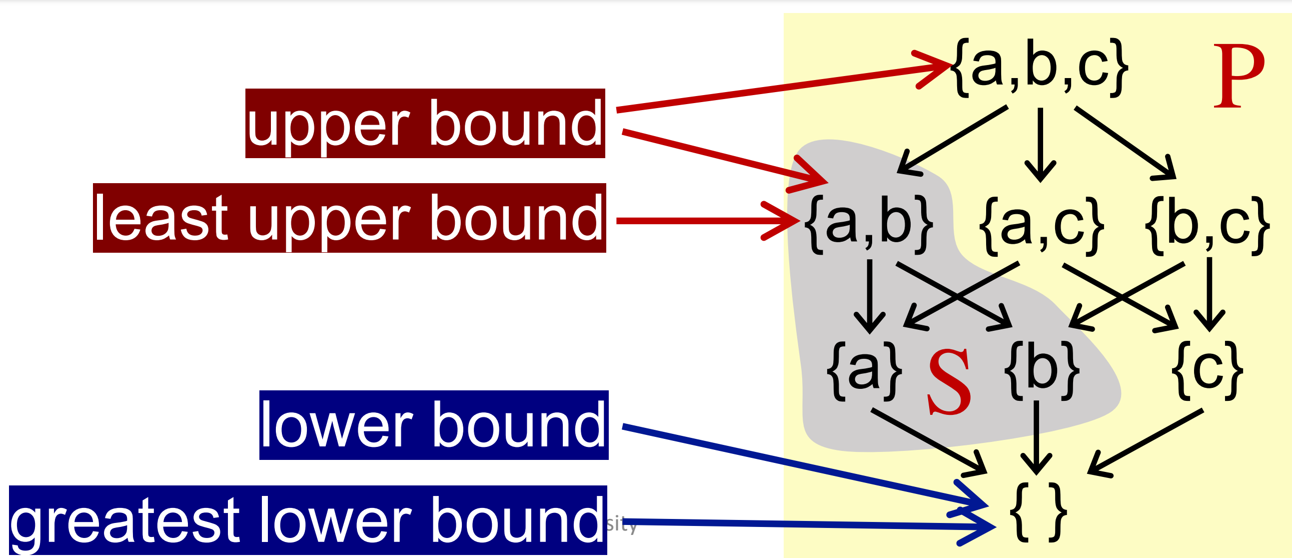

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Usually, if S contains only two elements a and b ($S = \{a, b\}$), then
$\sqcup S$ can be written $a \sqcup b$ (the join of a and b)
$\sqcap S$ can be written $a \sqcap b$ (the meet of a and b)

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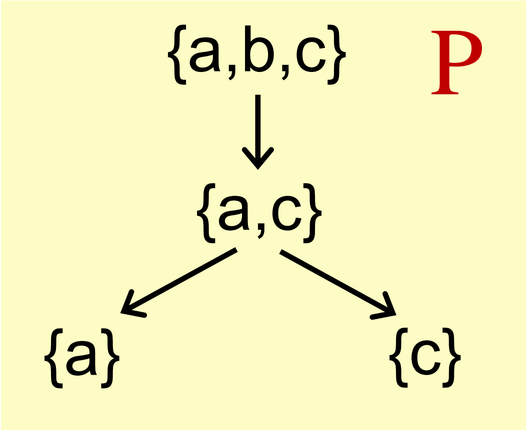

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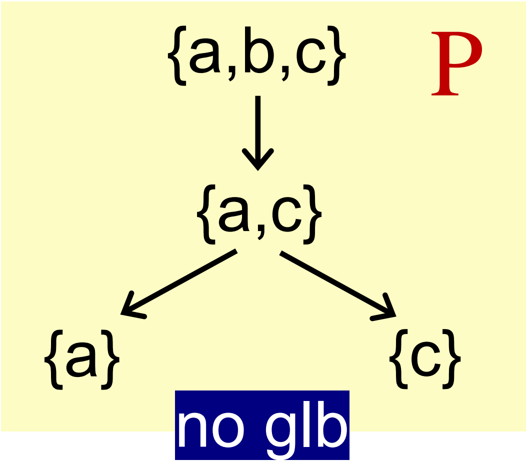

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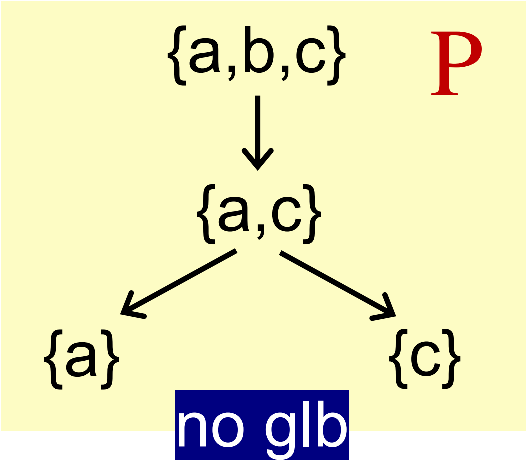

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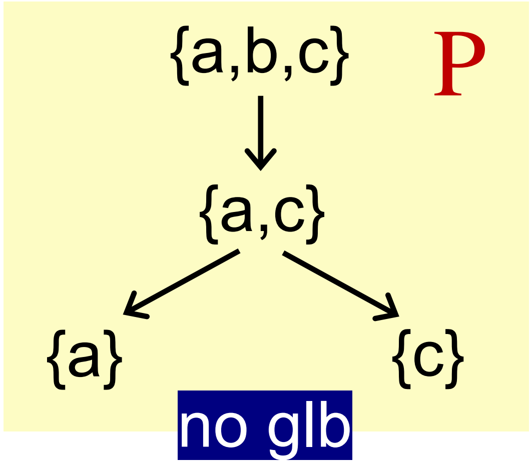

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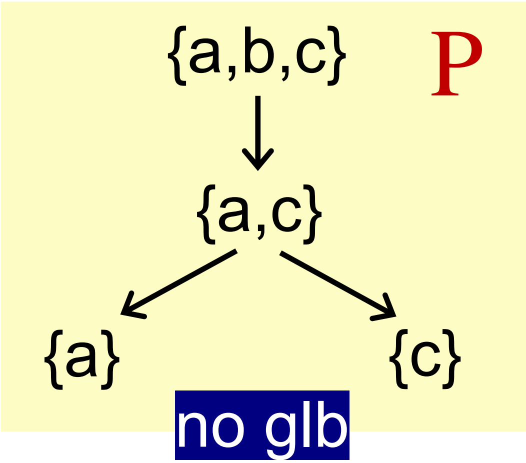

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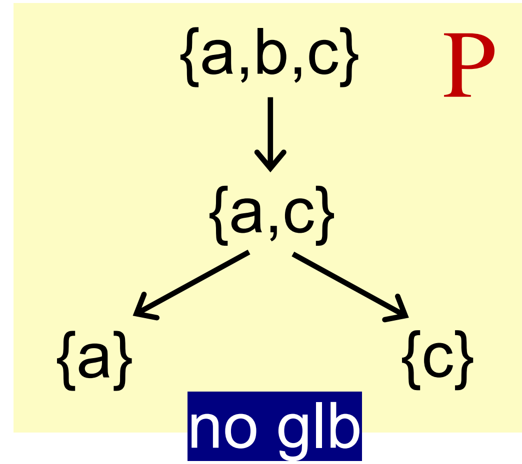

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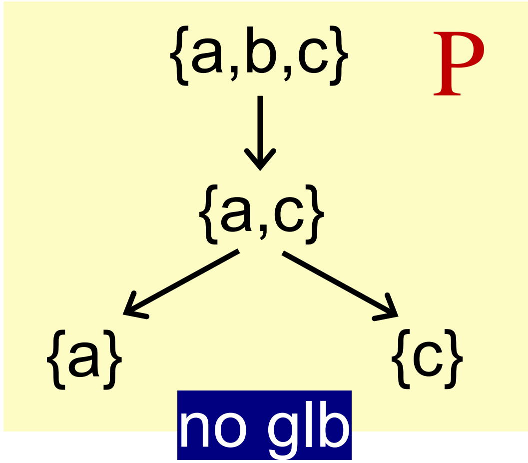

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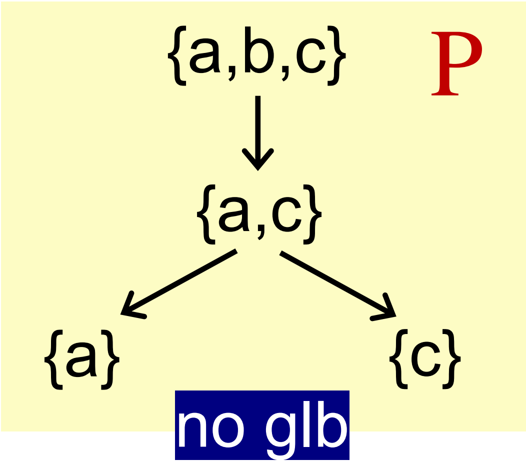

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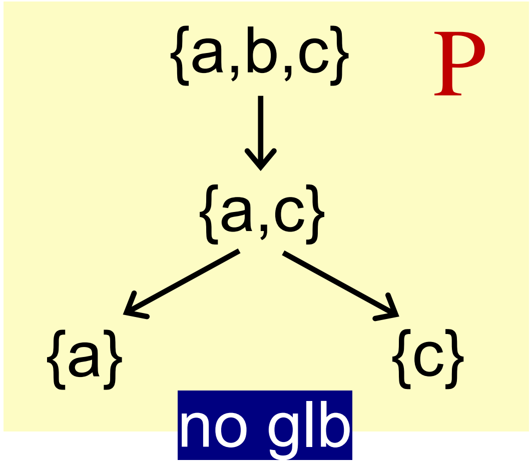

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$g_1 \sqsubseteq (g_2 = \sqcap P)$ and $g_2 \sqsubseteq (g_1 = \sqcap P)$

by the *antisymmetry* of partial order $\sqsubseteq$

$g_1 = g_2$

Lattice

Given a poset $(P, \sqsubseteq)$, $\forall a, b \in P$, if $a \sqcup b$ and $a \sqcap b$ exist, then $(P, \sqsubseteq)$ is called a lattice

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✓ The $\sqcup$ operator means "max" and $\sqcap$ operator means "min"

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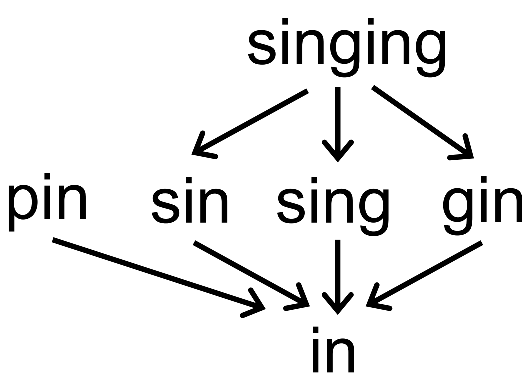

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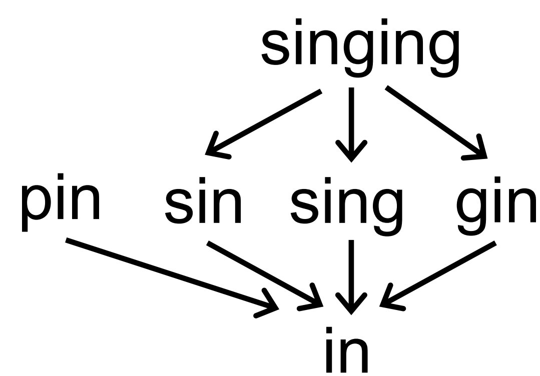

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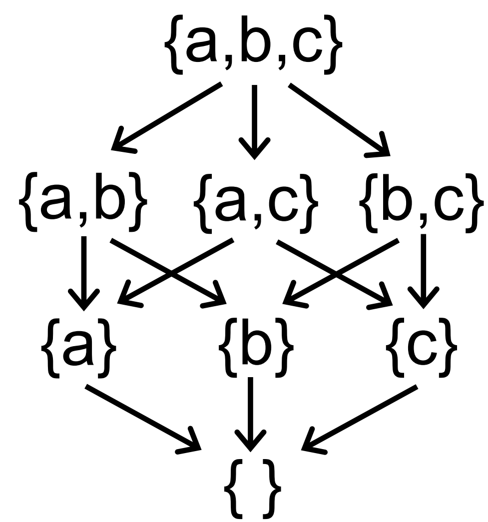

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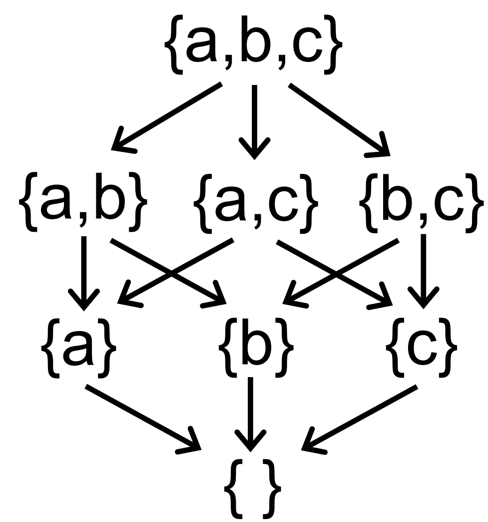
✓ The ⊔ operator means ∪ and ⊓ operator means ∩



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Semilattice

Given a poset $(P, \sqsubseteq)$, $\forall a, b \in P$,

if only $a \sqcup b$ exists, then $(P, \sqsubseteq)$ is called a join semilattice

if only $a \sqcap b$ exists, then $(P, \sqsubseteq)$ is called a meet semilattice

Complete Lattice

Given a lattice (P, $\sqsubseteq$), for arbitrary subset S of P, if $\sqcup S$ and $\sqcap S$ exist, then (P, $\sqsubseteq$) is called a complete lattice

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✗ For a subset S^+ including all positive integers, it has no $\sqcup S^+$ $(+\infty)$

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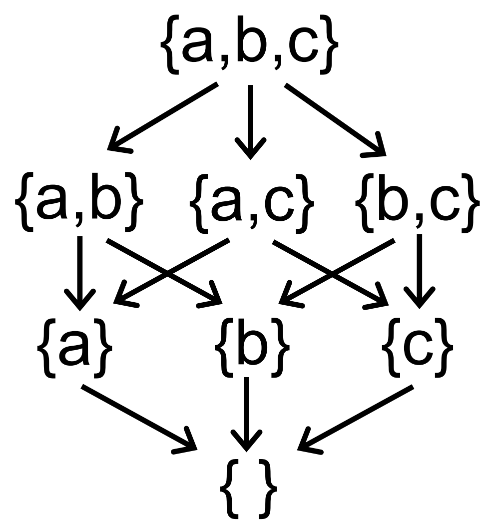

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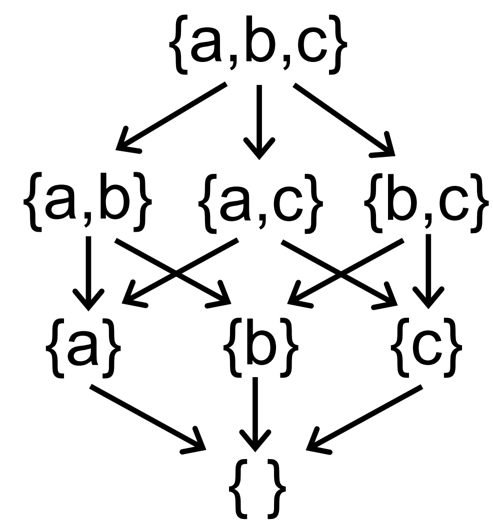
✓ Note: the definition of bounds implies that the bounds are not necessarily in the subsets (but they must be in the lattice)



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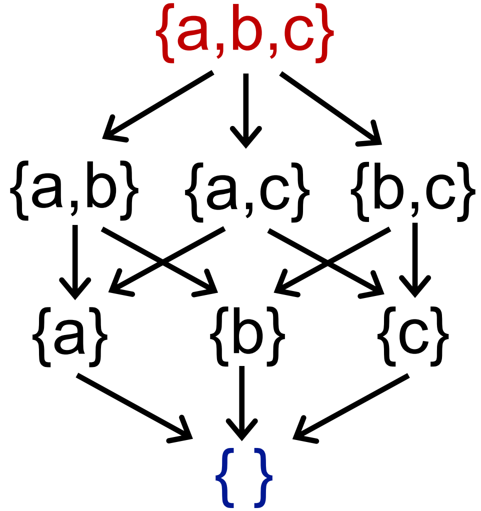
Every complete lattice (P, ⊑) has
a greatest element ⊤ = ⊔P called top and
a least element ⊥ = ⊓P called bottom



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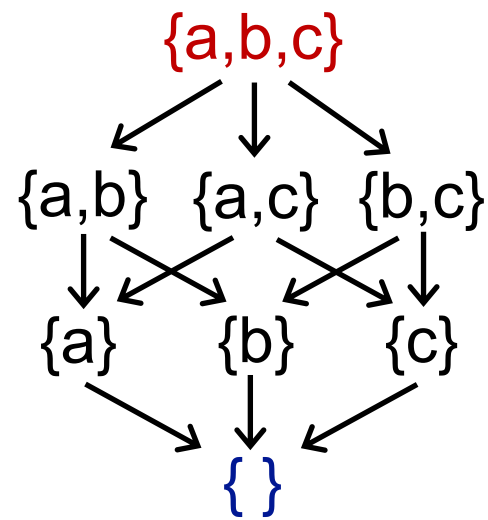

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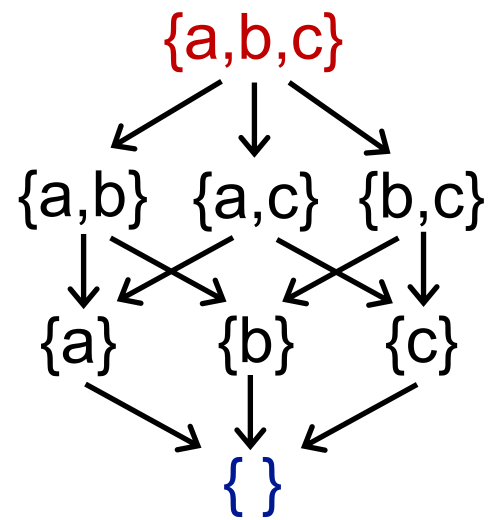
What about the opposite one?



Complete Lattice **Mostly focused in data flow analysis**

Given a lattice $(P, \sqsubseteq)$, for arbitrary subset S of P, if $\sqcup S$ and $\sqcap S$ exist, then $(P, \sqsubseteq)$ is called a complete lattice

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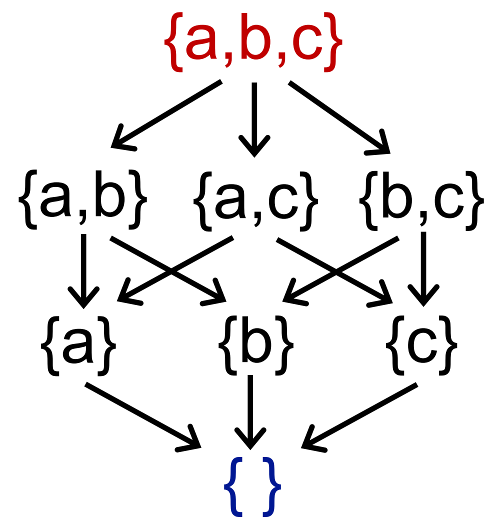

Product Lattice

Given lattices $L_1 = (P_1, \sqsubseteq_1)$, $L_2 = (P_2, \sqsubseteq_2)$, …, $L_n = (P_n, \sqsubseteq_n)$, if for all i, $(P_i, \sqsubseteq_i)$ has $\sqcup_i$ (least upper bound) and $\sqcap_i$ (greatest lower bound), then we can have a product lattice $L^n = (P, \sqsubseteq)$ that is defined by:

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- A product lattice is a lattice
- If a product lattice L is a product of complete lattices, then L is also complete

Data Flow Analysis Framework via Lattice

A data flow analysis framework (D, L, F) consists of:

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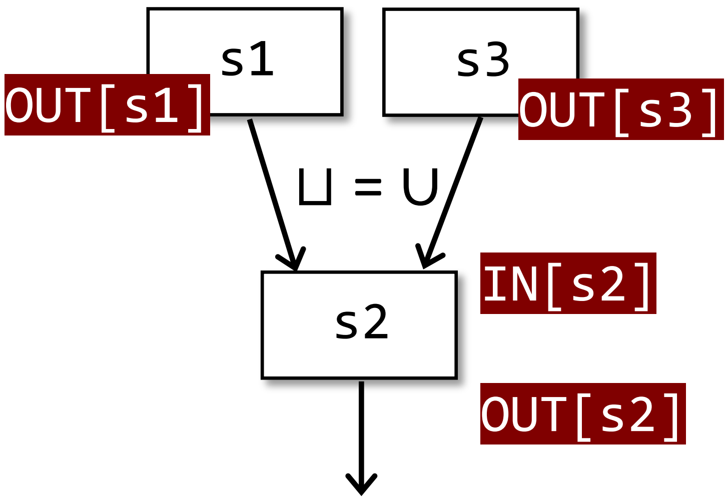

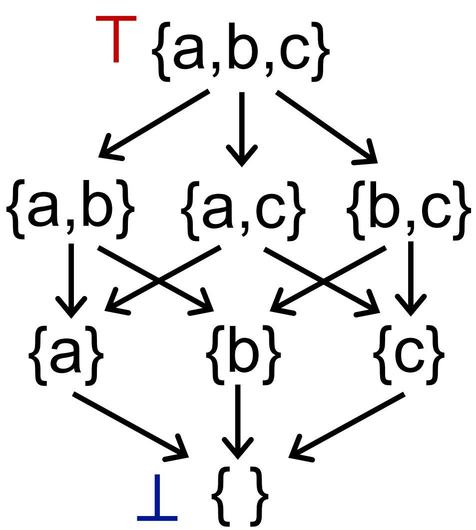

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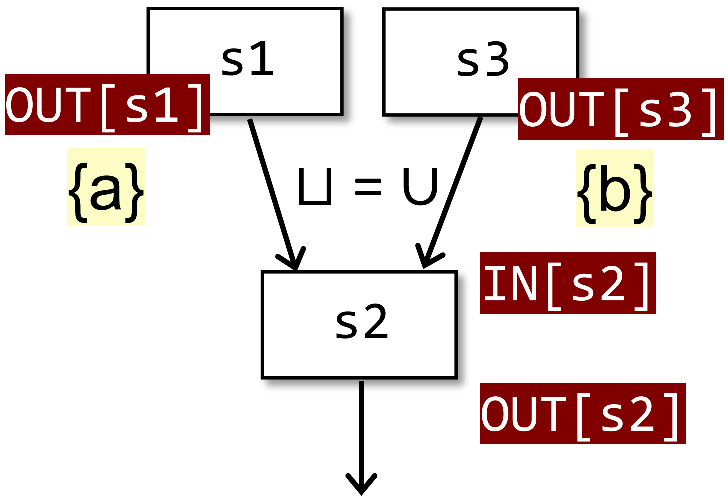

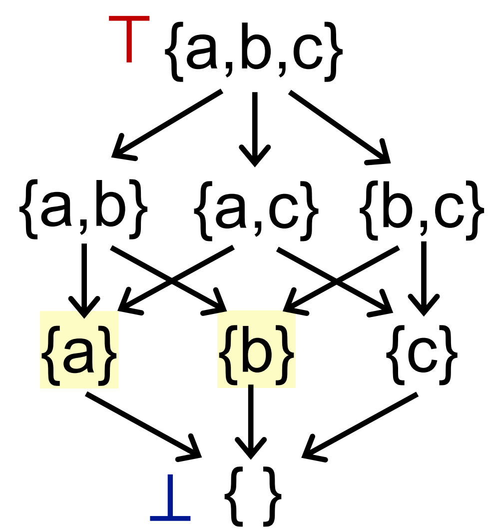

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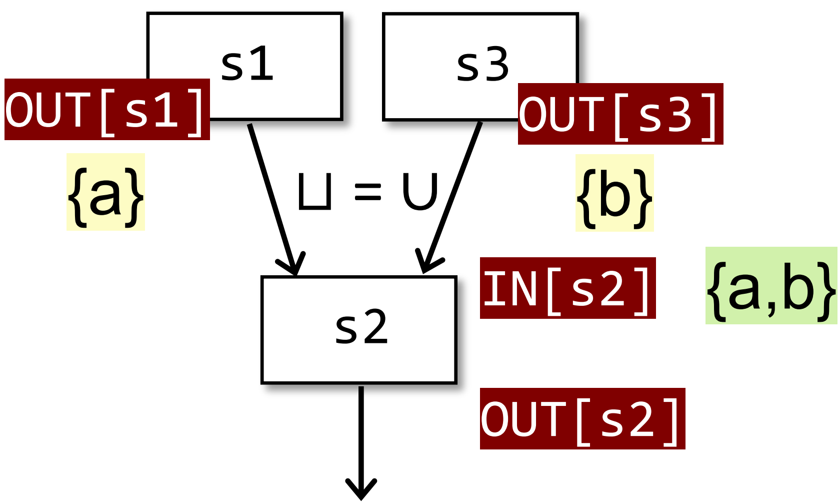

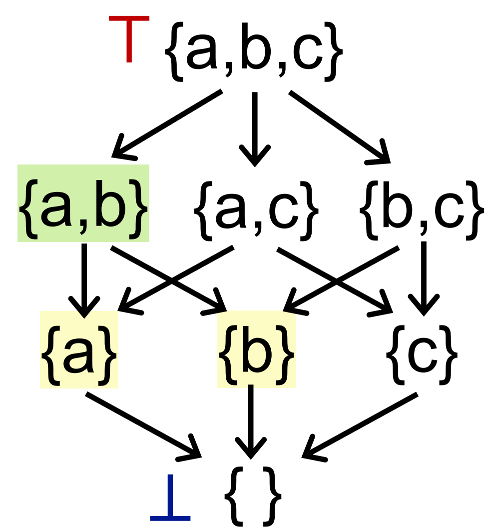

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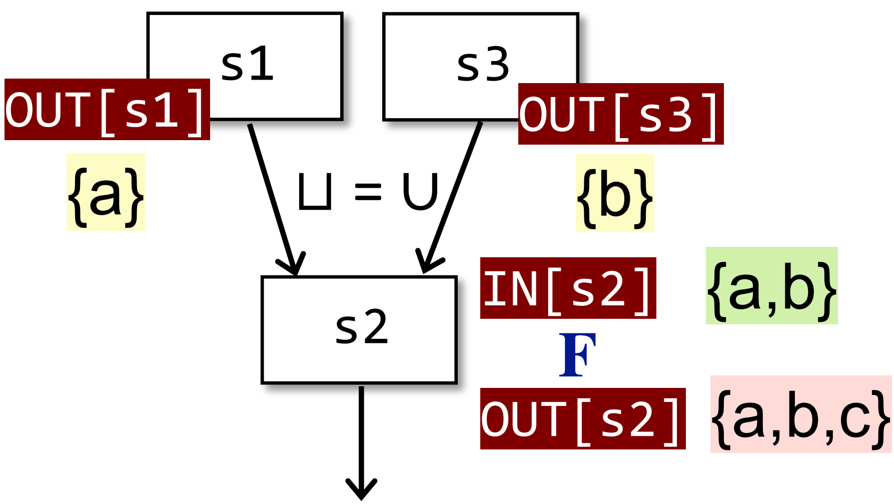

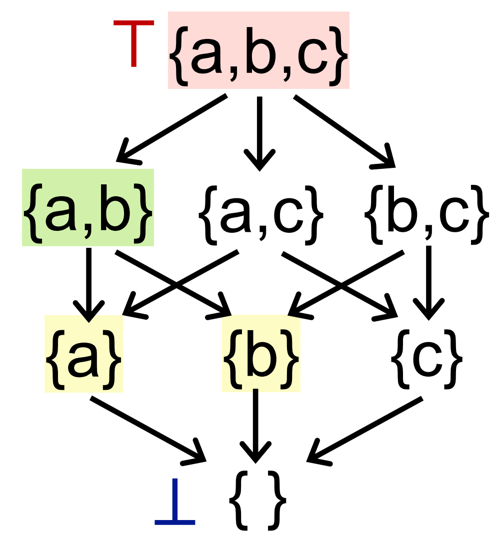

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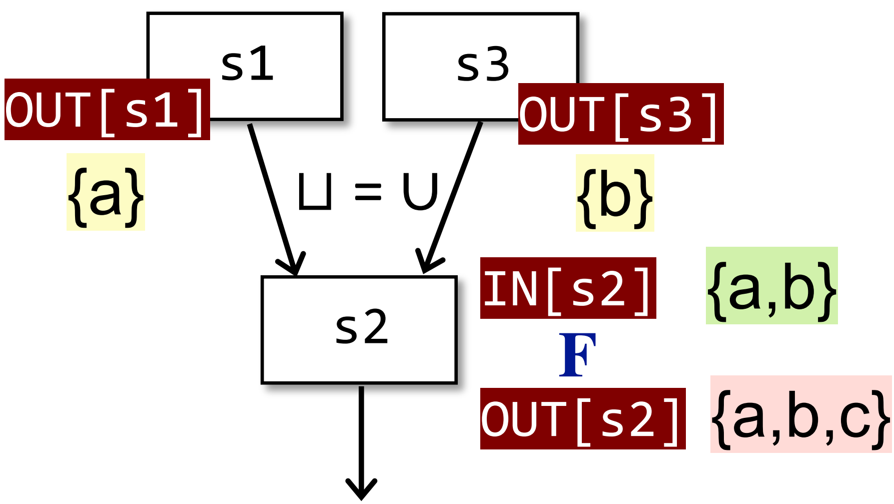

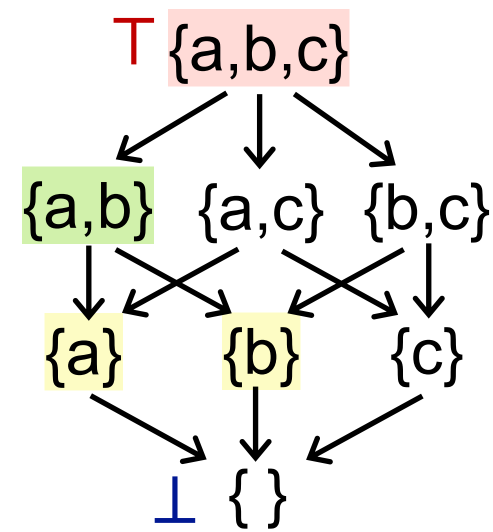

Data flow analysis can be seen as iteratively applying transfer functions and meet/join operations on the values of a lattice

Review The Questions We Have Seen Before

The iterative algorithm (or the IN/OUT equation system) produces a solution to a data flow analysis

- Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?
- If so, is there only one solution or only one fixed point? If more than one, is our solution the best one (most precise)?
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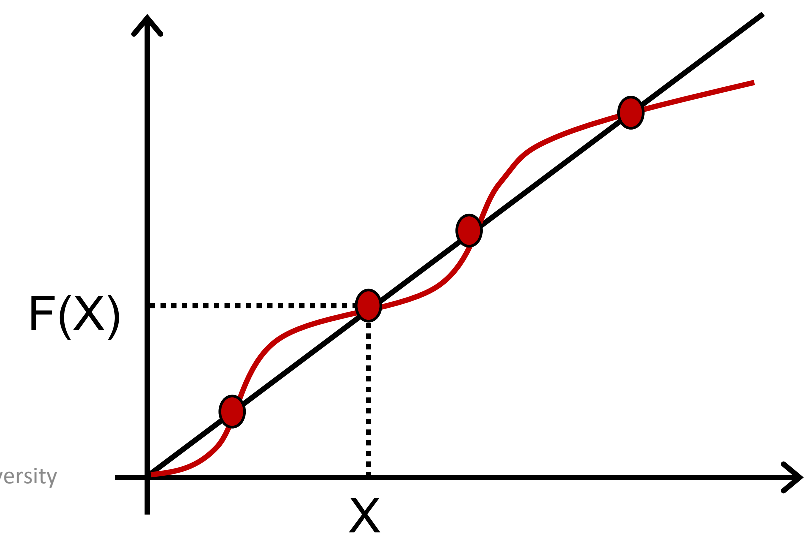

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A function f: L → L (L is a lattice) is monotonic if $\forall x, y \in L$,

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the least fixed point of f can be found by iterating

$f(\perp), f(f(\perp)), \ldots, f^k(\perp)$ until a fixed point is reached

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the least fixed point of f can be found by iterating

$f(\bot), f(f(\bot)), \ldots, f^k(\bot)$ until a fixed point is reached

the greatest fixed point of f can be found by iterating

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> $f(\bot), f(f(\bot)), \ldots, f^k(\bot)$ until a fixed point is reached
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> $f(\top$ …, $f^k(\top)$ until a fixed point is reached

Let us prove

(1) Existence of fixed point

(2) The fixed point is the least

Fixed-Point Theorem (Existence of Fixed Point)

Proof:

By the definition of $\perp$ and $f: L \rightarrow L$, we have

$$\perp \sqsubseteq f(\perp)$$

Fixed-Point Theorem (Existence of Fixed Point)

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As f is monotonic, we have

$$f(\perp) \sqsubseteq f(f(\perp)) = f^2(\perp)$$

Fixed-Point Theorem (Existence of Fixed Point)

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Similarly (by repeatedly applying f), we have

$$\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \ \ldots \sqsubseteq f^i(\perp)$$

Fixed-Point Theorem (Existence of Fixed Point)

Proof:

By the definition of $\perp$ and f: $L \rightarrow L$, we have

$$\perp \sqsubseteq f(\perp)$$

As f is monotonic, we have

$$f(\perp) \sqsubseteq f(f(\perp)) = f^2(\perp)$$

Similarly (by repeatedly applying f), we have

$$\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \ \dots \sqsubseteq f^i(\perp)$$

As L is finite, for some k, we have

$$f^{Fix} = f^k(\perp) = f^{k+1}(\perp)$$

Thus, the fixed point exists.

Fixed-Point Theorem (Least Fixed Point)

Proof:
Assume we have another fixed point x, i.e., $x = f(x)$

Fixed-Point Theorem (Least Fixed Point)

Proof:

Assume we have another fixed point x, i.e., x = f(x)

By the definition of $\perp$, we have $\perp \sqsubseteq x$

Fixed-Point Theorem (Least Fixed Point)

Proof:

Assume we have another fixed point x, i.e., $x = f(x)$

By the definition of $\bot$, we have $\bot \sqsubseteq x$

Induction begins:

Fixed-Point Theorem (Least Fixed Point)

Proof:

Assume we have another fixed point x, i.e., $x = f(x)$

By the definition of $\perp$, we have $\perp \sqsubseteq x$

Induction begins:

As f is monotonic, we have

$$f(\perp) \sqsubseteq f(x)$$

Fixed-Point Theorem (Least Fixed Point)

Proof:

Assume we have another fixed point x, i.e., $x = f(x)$

By the definition of $\perp$, we have $\perp \sqsubseteq x$

Induction begins:

As f is monotonic, we have

$$f(\perp) \sqsubseteq f(x)$$

Assume $f^i(\perp) \sqsubseteq f^i(x)$, as f is monotonic, we have

$$f^{i+1}(\perp) \sqsubseteq f^{i+1}(x)$$

Fixed-Point Theorem (Least Fixed Point)

Proof:

Assume we have another fixed point x, i.e., $x = f(x)$

By the definition of $\perp$, we have $\perp \sqsubseteq x$

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Assume $f^i(\perp) \sqsubseteq f^i(x)$, as f is monotonic, we have

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Thus by induction, we have

$$f^i(\perp) \sqsubseteq f^i(x)$$

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Thus by induction, we have

$$f^i(\perp) \sqsubseteq f^i(x)$$

Thus $f^i(\perp) \sqsubseteq f^i(x) = x$, then we have

$$f^{Fix} = f^k(\perp) \sqsubseteq x$$

Thus the fixed point is the least

Fixed-Point Theorem (Least Fixed Point)

Proof:

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By the definition of $\perp$, we have $\perp \sqsubseteq x$

Induction begins:

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Thus the fixed point is the least

The proof for greatest fixed point is similar

Fixed-Point Theorem

Given a complete lattice $(L, \sqsubseteq)$, if

(1) $f: L \rightarrow L$ is monotonic and (2) L is finite, then

the least fixed point of f can be found by iterating

$f(\bot), f(f(\bot)), \ldots, f^k(\bot)$ until a fixed point is reached

the greatest fixed point of f can be found by iterating

$f(\top), f(f(\top)), \ldots, f^k(\top)$ until a fixed point is reached

Review The Questions We Have Seen Before

The iterative algorithm (or the IN/OUT equation system) produces a solution to a data flow analysis

- Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?
- If so, is there only one solution or only one fixed point? If more than one, is our solution the best one (most precise)?
- When will the algorithm reach the fixed point, or when can we get the solution?

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- When will the algorithm reach the fixed p can we get the solution?

greatest or least fixed point

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- Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?
- If so, is there only one solution or only one fixed point? If more than one, is our solution the best one (most [illegible]?
- When will the algorithm reach the fixed [illegible] can we get the solution?

greatest or least fixed point

Now what we have just seen is the property (fixed point theorem) for the function on a lattice. We cannot say our iterative algorithm also has that property unless we can *relate the algorithm to the fixed point theorem*, if possible

Relate Iterative Algorithm to Fixed-Point Theorem

$\rightarrow (\perp, \perp, \ldots, \perp)$

iter 1 $\rightarrow (v_1^1, v_2^1, \ldots, v_k^1)$

iter 2 $\rightarrow (v_1^2, v_2^2, \ldots, v_k^2)$

$\vdots$

iter i $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$

iter i+1 $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$

$\Downarrow$

Given a complete lattice $(L, \sqsubseteq)$, if

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the greatest fixed point of f can be found by iterating

$f(\top), f(f(\top)), \ldots, f^k(\top)$ until a fixed point is reached

Relate Iterative Algorithm to Fixed-Point Theorem

(L, L, …, L)

$$\rightarrow (\bot, \ \bot, \ \ldots, \ \bot)$$

iter 1 $\rightarrow (v_1^1, v_2^1, \ldots, v_k^1)$

iter 2 $\rightarrow (v_1^2, v_2^2, \ldots, v_k^2)$

$\vdots$

iter i $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$

iter i+1 $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$

If a product lattice L^k is a product of complete (and finite) lattices, i.e., (L, L, …, L), then L^k is also complete (and finite)

$\Downarrow$

Given a complete lattice (L, ⊑), if

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the greatest fixed point of f can be found by iterating

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Relate Iterative Algorithm to Fixed-Point Theorem

(L, L, …, L)

$$(\bot,\ \bot,\ \dots,\ \bot)$$

iter 1 $\rightarrow (v_1^1, v_2^1, \dots, v_k^1)$

iter 2 $\rightarrow (v_1^2, v_2^2, \dots, v_k^2)$

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In each iteration, it is equivalent to think that we apply function F which consists of

(1) transfer function f_i: L → L for every node
(2) join/meet function ⊔/⊓: L×L → L for control-flow confluence

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$f(\top), f(f(\top)), \dots, f^k(\top)$ until a fixed point is reached

Relate Iterative Algorithm to Fixed-Point Theorem

$$(L, L, \ldots, L)$$

$$\rightarrow (\bot, \bot, \ldots, \bot)$$

iter 1 $\rightarrow (v_1^1, v_2^1, \ldots, v_k^1)$

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Now the remaining issue is to prove that function F is monotonic

Prove Function F is Monotonic

In each iteration, it is equivalent to think that we apply function F which consists of

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Gen/Kill function is monotonic

Prove Function F is Monotonic

In each iteration, it is equivalent to think that we apply function F which consists of
(1) transfer function f_i: $L \rightarrow L$ for every node
(2) join/meet function $\sqcup / \sqcap$: $L \times L \rightarrow L$ for control-flow confluence

Actually the binary operator is a basic case of $L \times L \times \ldots \times L$,

Gen/Kill function is monotonic

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We want to show that $\sqcup$ is monotonic

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Actually the binary operator is a basic case of L × L ×… × L,

We want to show that ⊔ is monotonic

Proof.

$\forall x, y, z \in L, x \sqsubseteq y$, we want to prove $x \sqcup z \sqsubseteq y \sqcup z$

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by the definition of ⊔, $y \sqsubseteq y \sqcup z$

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$\forall x, y, z \in L, x \sqsubseteq y$, we want to prove $x \sqcup z \sqsubseteq y \sqcup z$

by the definition of $\sqcup$, $y \sqsubseteq y \sqcup z$

by transitivity of $\sqsubseteq$, $x \sqsubseteq y \sqcup z$

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Proof.
$\forall x, y, z \in L$, $x \sqsubseteq y$, we want to prove $x \sqcup z \sqsubseteq y \sqcup z$
by the definition of ⊔, $y \sqsubseteq y \sqcup z$
by transitivity of ⊑, $x \sqsubseteq y \sqcup z$
thus $y \sqcup z$ is an upper bound for x, and also for z (by ⊔'s definition)

Prove Function F is Monotonic

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as $x \sqcup z$ is the least upper bound of x and z

Prove Function F is Monotonic

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We want to show that ⊔ is monotonic

Proof.

∀x, y, z ∈ L, x ⊑ y, we want to prove x ⊔ z ⊑ y ⊔ z

by the definition of ⊔, y ⊑ y ⊔ z

by transitivity of ⊑, x ⊑ y ⊔ …

thus y ⊔ z is an upp… (by ⊔'s definition)

as x ⊔ z is the least …ound of x and z

thus x ⊔ z ⊑ y ⊔ z

Thus the fixed point theorem applies to the iterative algorithm for data flow analysis

Review The Questions We Have Seen Before

The iterative algorithm (or the IN/OUT equation system) produces a solution to a data flow analysis

- Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?
- If so, is there only one solution or only one fixed point? If more than one, is our solution the best one (m…

greatest or least fixed point

- When will the algorithm reach the fixed point, or when can we get the solution?

Now what we have just seen is the property (fixed point theorem) for the function on a lattice. ~~We cannot say our iterative algorithm also has that property unless~~ we can relate the algorithm to the fixed point theorem, ~~if possible~~

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When Will the Algorithm Reach the Fixed Point?

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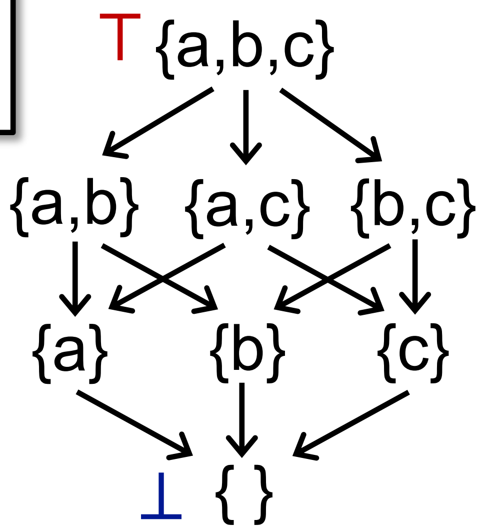
The height of a lattice *h* is the length of the longest path from Top to Bottom in the lattice.



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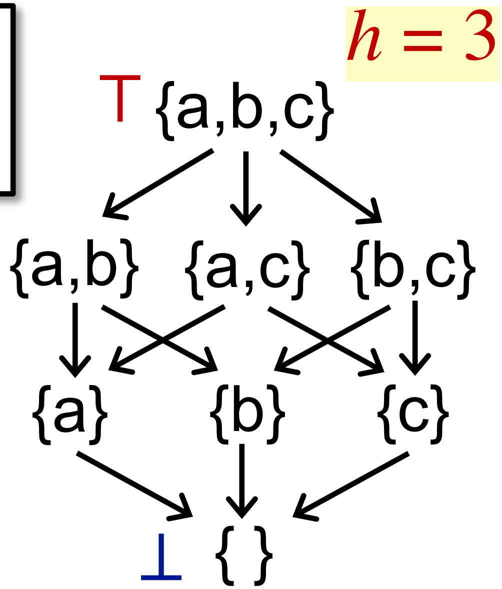
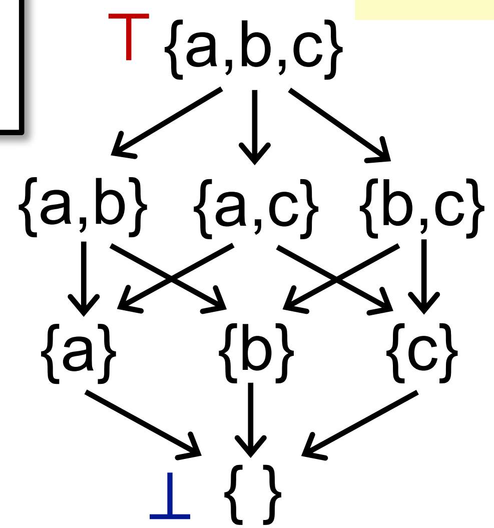


When Will the Algorithm Reach the Fixed Point?

The height of a lattice h is the length of the longest path from Top to Bottom in the lattice.

The maximum iterations i needed to reach the fixed point

$h = 3$



$$\rightarrow (\bot, \ \bot, \ \ldots, \ \bot)$$

iter 1 $\rightarrow (v_1^1, v_2^1, \ldots, v_k^1)$

iter 2 $\rightarrow (v_1^2, v_2^2, \ldots, v_k^2)$

$$\vdots$$

iter i $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$

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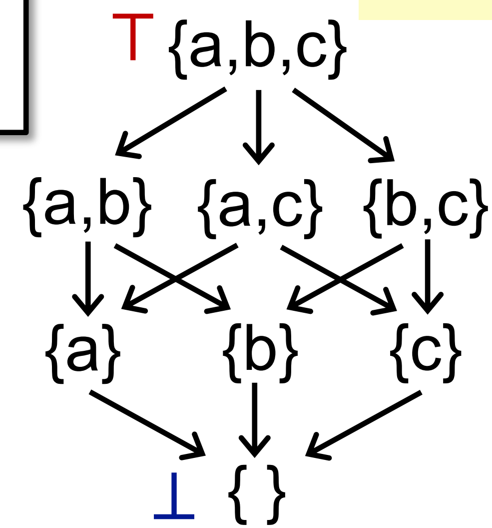

The maximum iterations i needed to reach the fixed point

$$(\perp, \ \perp, \ \ldots, \ \perp)$$

iter 1 $(v_1^1, v_2^1, \ldots, v_k^1)$

iter 2 $(v_1^2, v_2^2, \ldots, v_k^2)$

$\vdots$

iter i $(v_1^i, v_2^i, \ldots, v_k^i)$

iter i+1 $(v_1^i, v_2^i, \ldots, v_k^i)$

In each iteration, assume only one step in the lattice (upwards or downwards) is made in one node (e.g., one 0->1 in RD)

When Will the Algorithm Reach the Fixed Point?

The height of a lattice h is the length of the longest path from Top to Bottom in the lattice.

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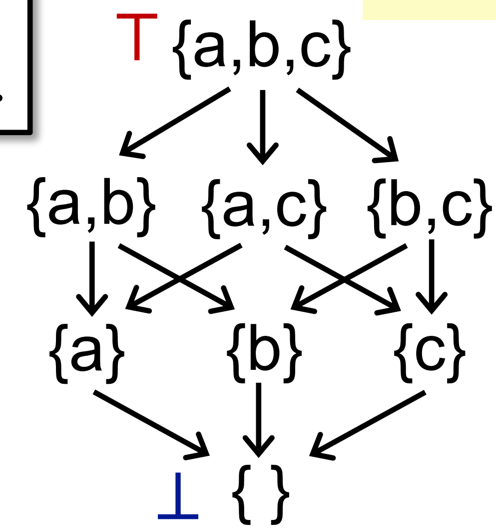

The maximum iterations i needed to reach the fixed point

$$\rightarrow (\perp, \ \perp, \ \ldots, \ \perp)$$

$$\textit{iter } 1 \rightarrow (v_1^1, v_2^1, \ldots, v_k^1)$$

$$\textit{iter } 2 \rightarrow (v_1^2, v_2^2, \ldots, v_k^2)$$

$$\vdots$$

$$\textit{iter } i \rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$$

$$\textit{iter } i{+}1 \rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$$

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Assume the lattice height is h and the number of nodes in CFG is k

When Will the Algorithm Reach the Fixed Point?

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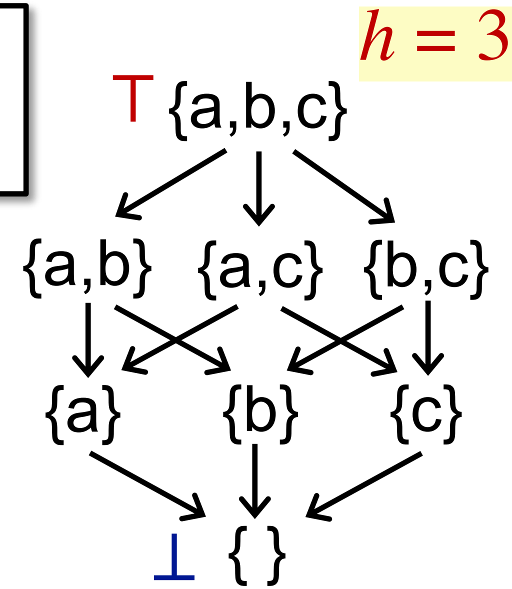

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$$\rightarrow (\bot, \ \bot, \ \ldots, \ \bot)$$

iter 1 $\rightarrow (v_1^1, v_2^1, \ldots, v_k^1)$

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$\vdots$

iter i $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$

iter i+1 $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$

In each iteration, assume only one step in the lattice (upwards or downwards) is made in one node (e.g., one 0->1 in RD)

Assume the lattice height is h and the number of nodes in CFG is k

We need at most $i = h*k$ iterations

Review The Questions We Have Seen Before

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YES

- Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?

- If so, ~~is there only one solution or only one fixed point~~? If more than one, is our solution the best one (most precise)?

YES

- When will the algorithm reach the fixed point, or when can we get the solution?

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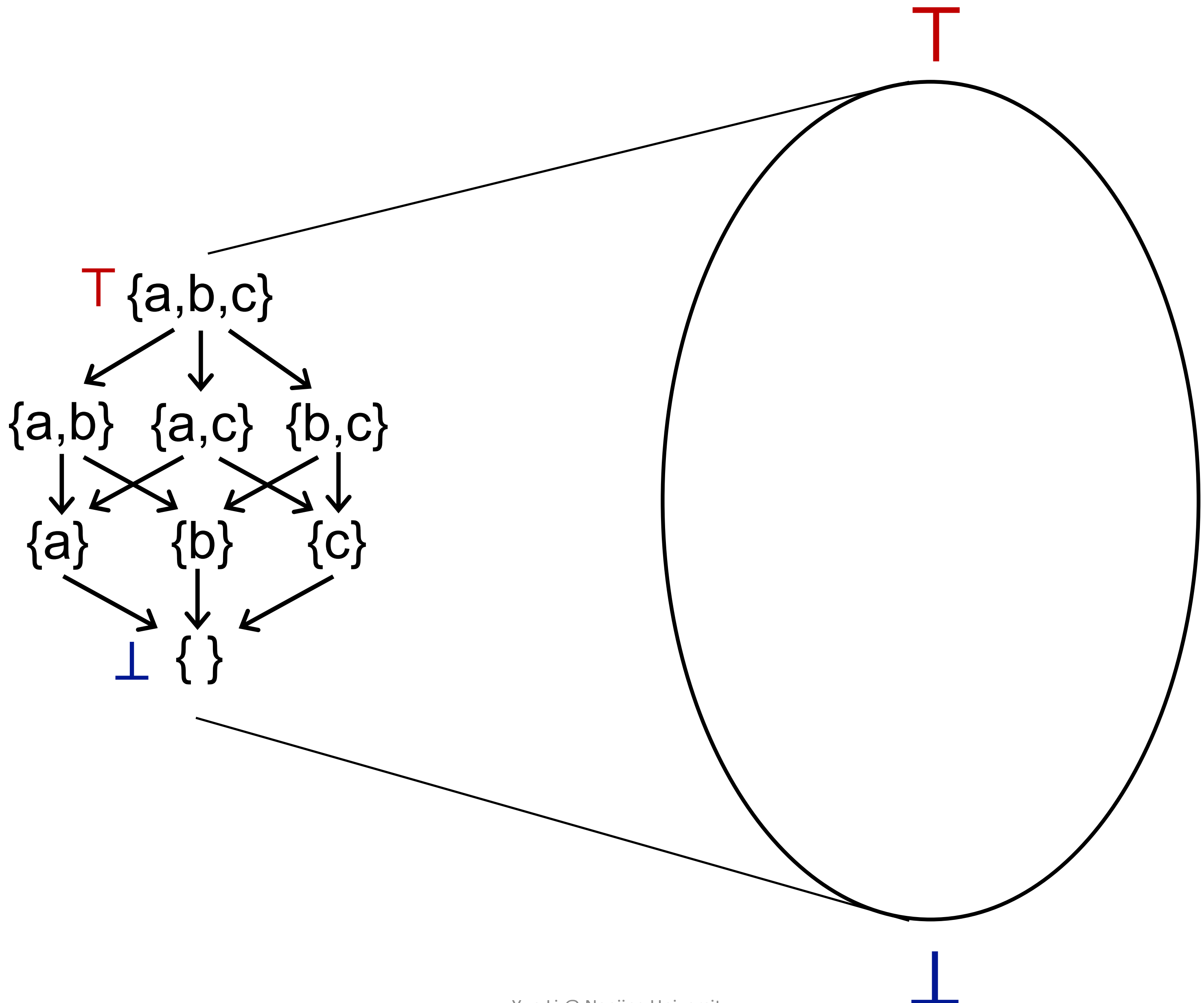
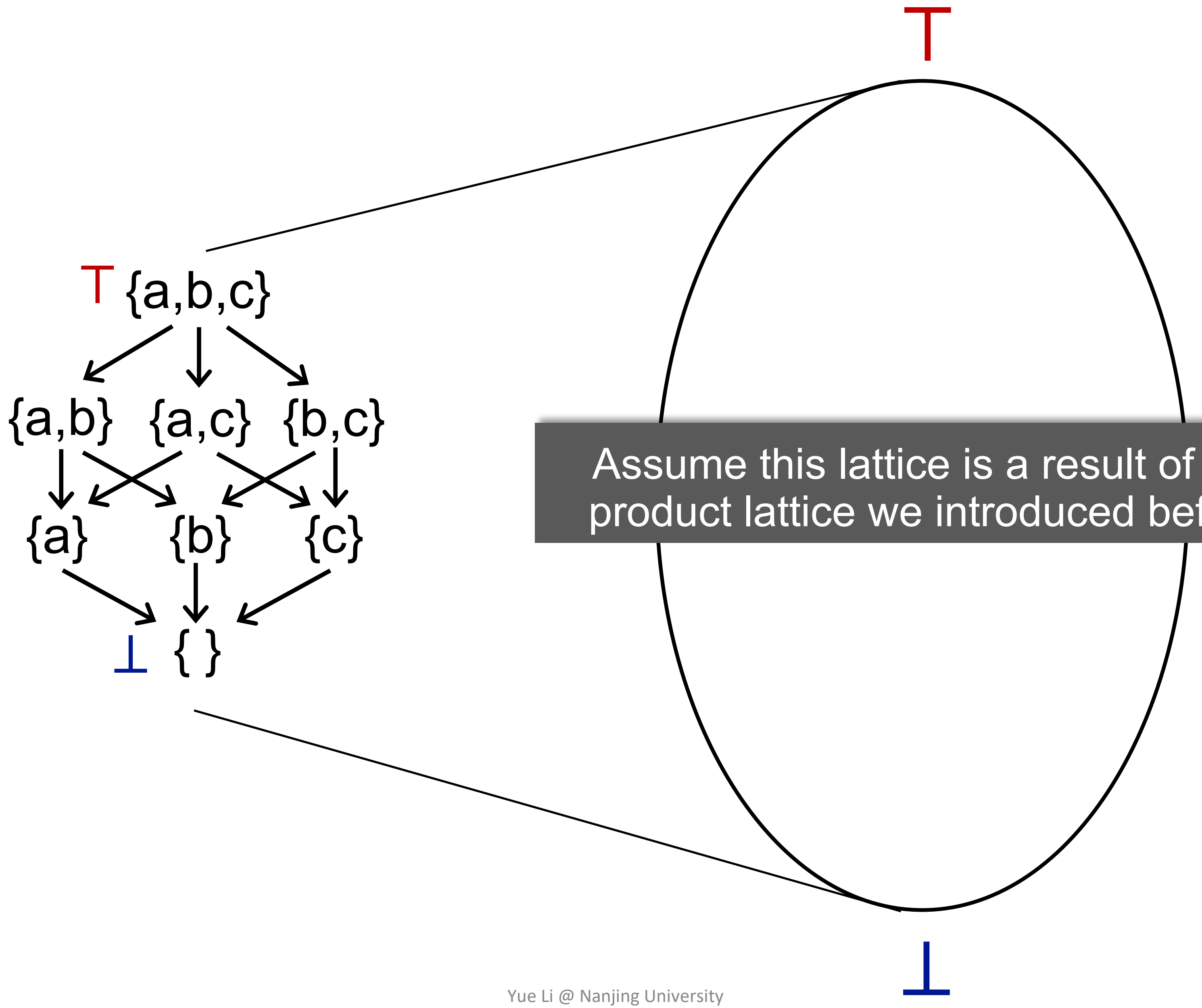
YES

- When will the algorithm reach the fixed point, or when can we get the solution?

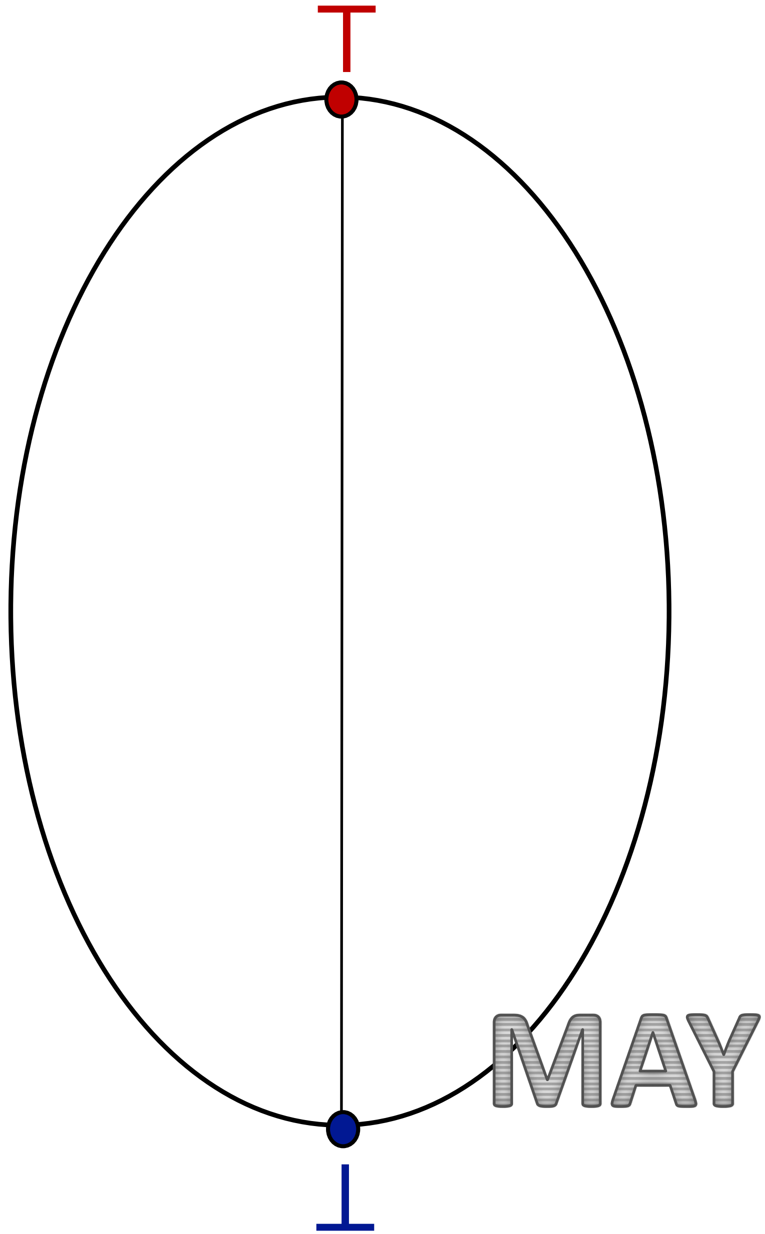
Worst case of #iterations:
the product of the lattice height and the number of nodes in CFG

May and Must Analyses, a Lattice View

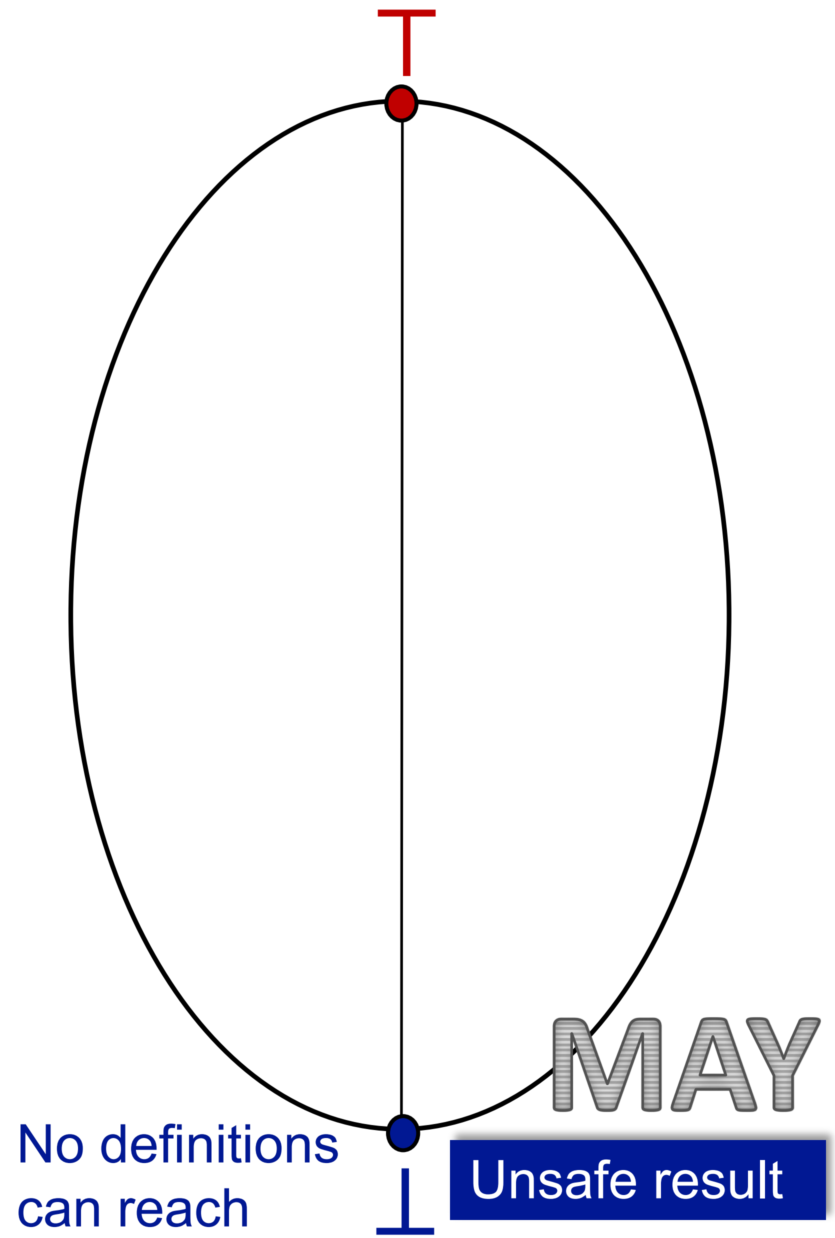

⊤
⊤ {a,b,c}
{a,b} {a,c} {b,c}
{a} {b} {c}
⊥ { }
⊥



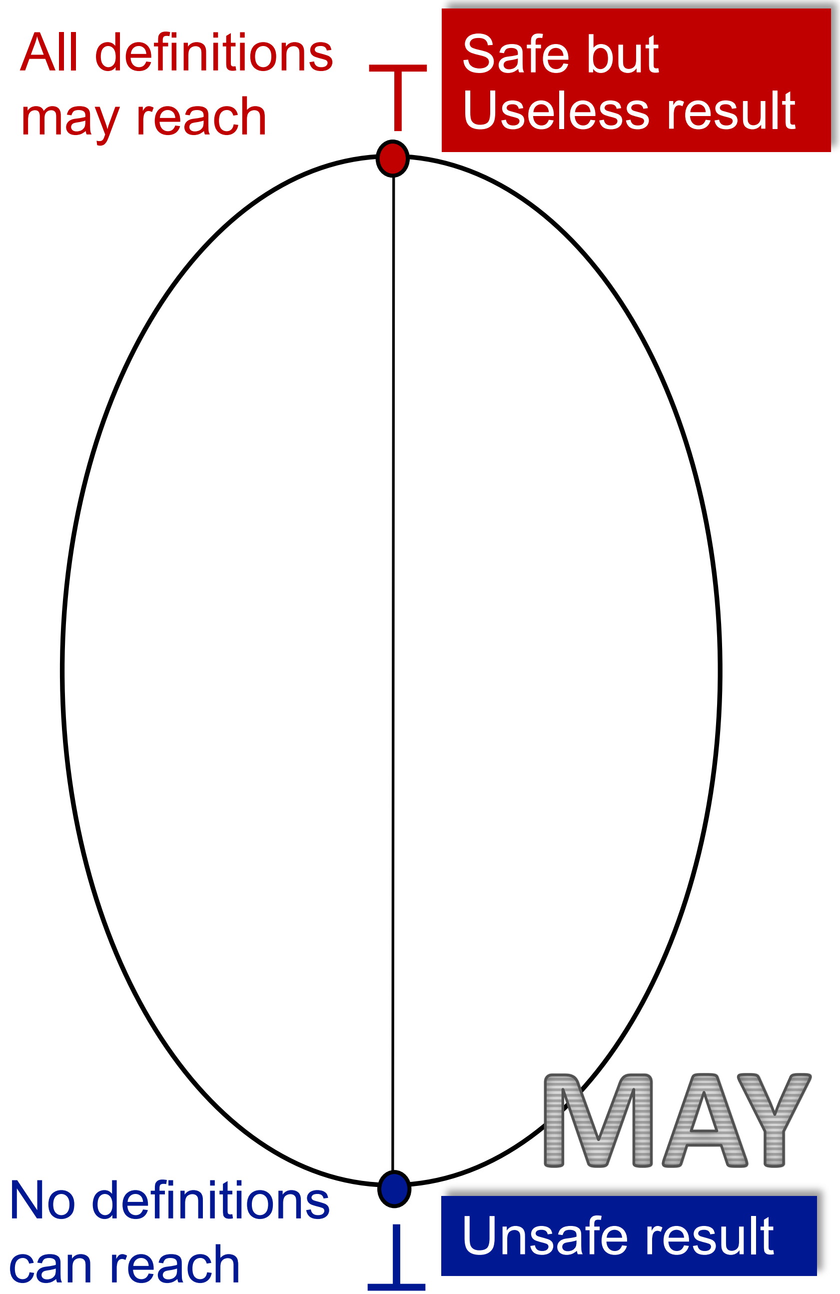
Assume this lattice is a result of the product lattice we introduced before

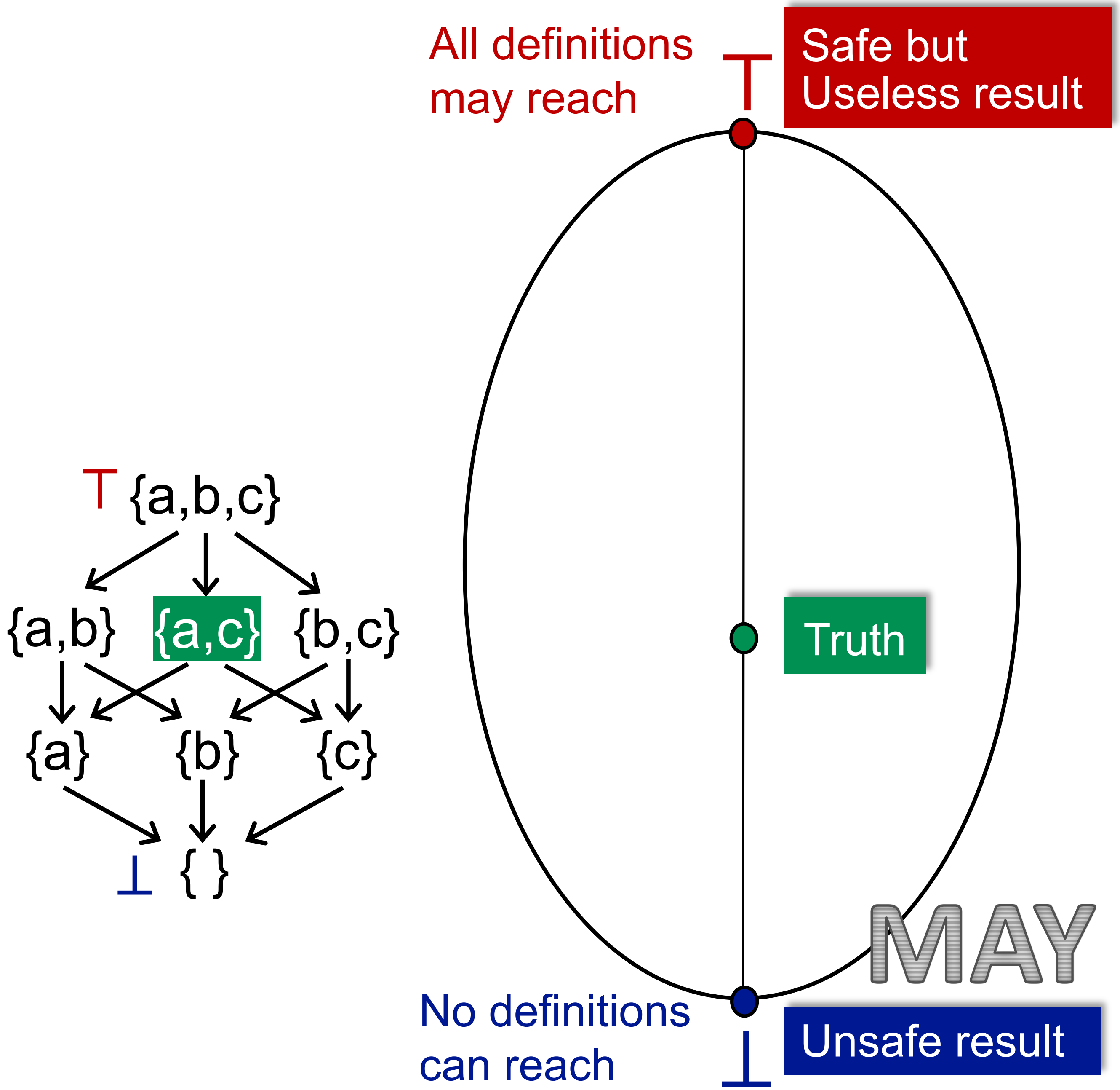


MAY



⊤
MAY
No definitions
can reach
⊥
Unsafe result


All definitions
may reach
Safe but
Useless result
MAY
No definitions
can reach
Unsafe result


⊤ {a,b,c}
{a,b}
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{b,c}
{a}
{b}
{c}
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All definitions may reach
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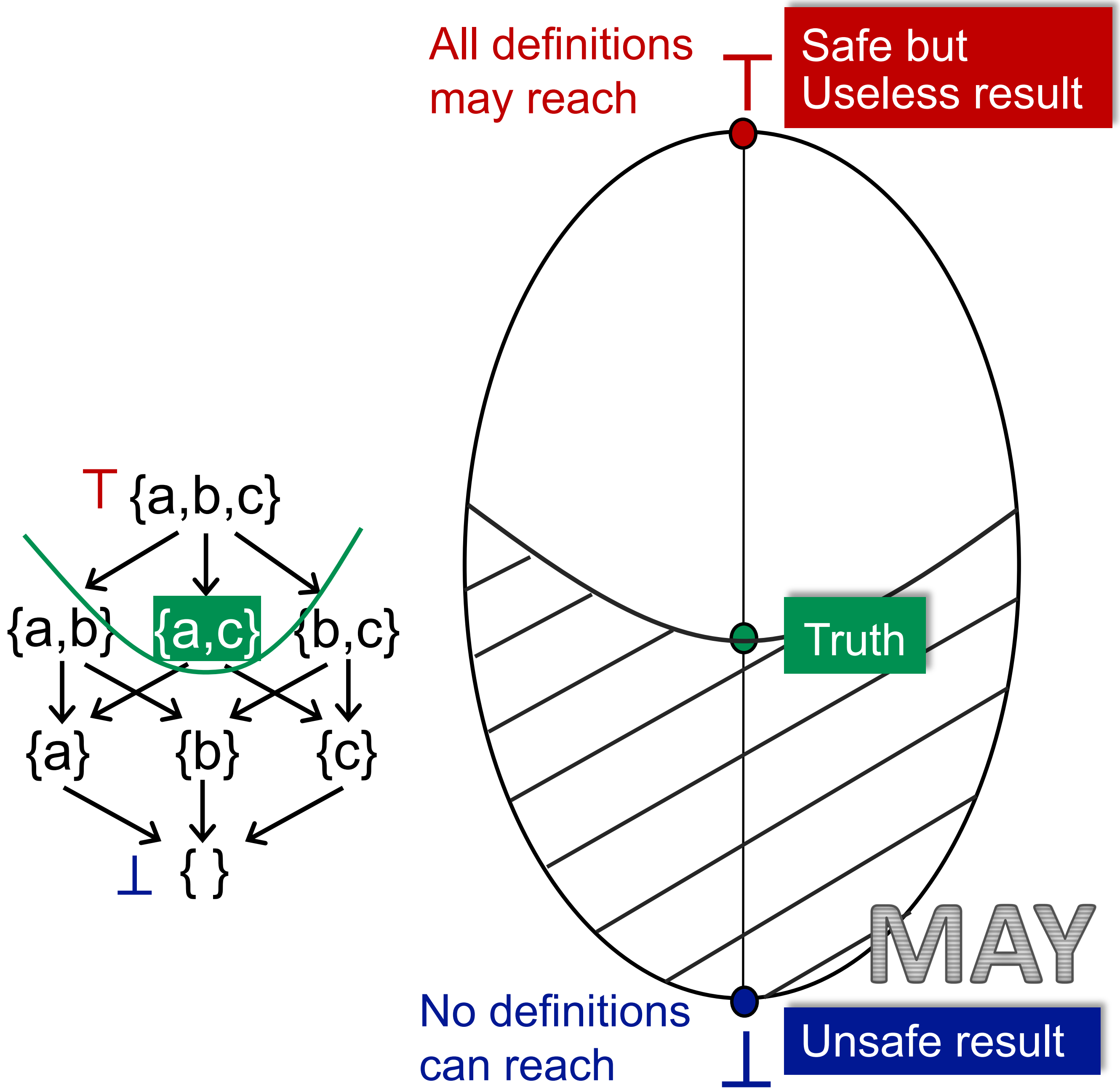

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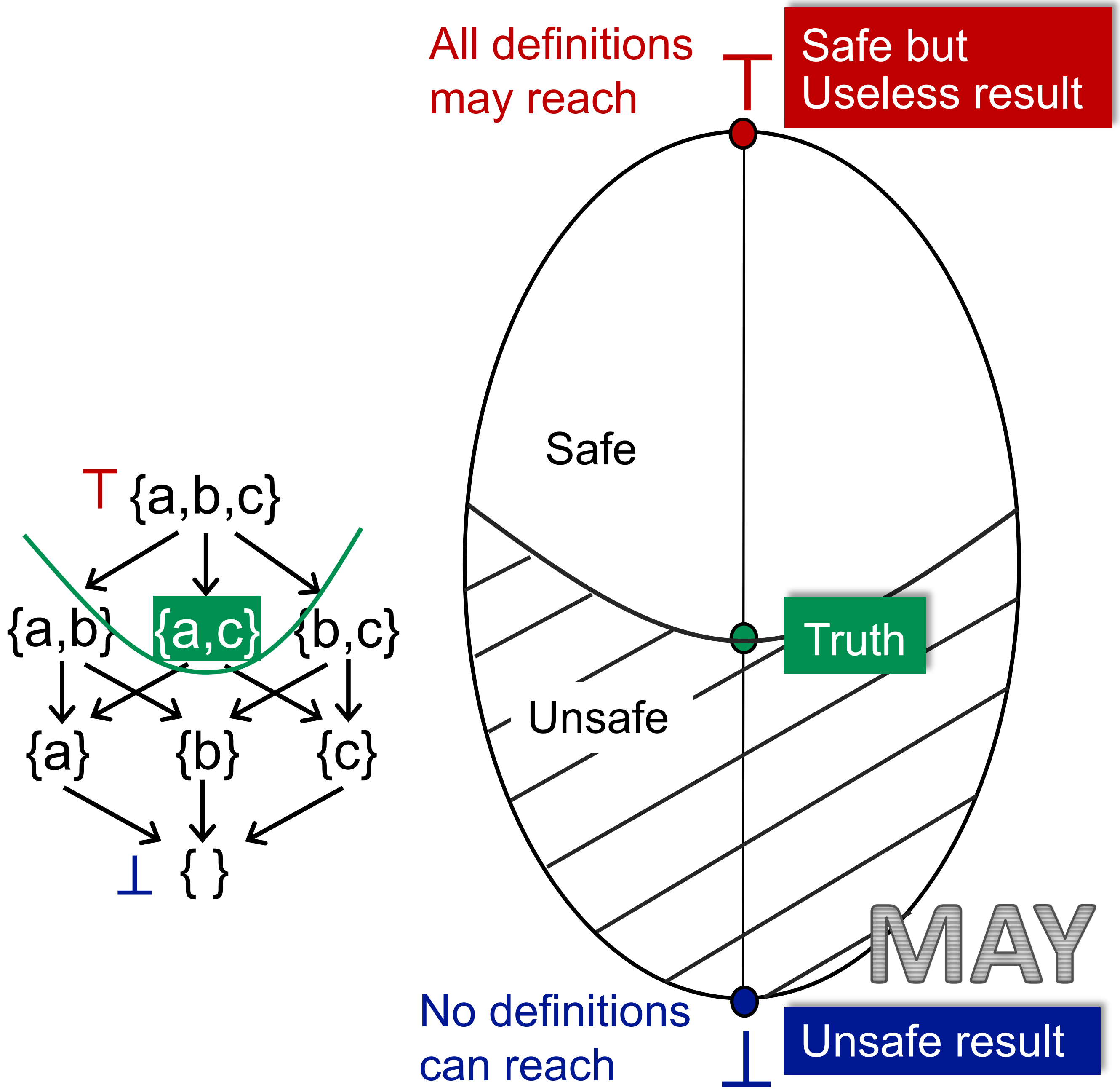
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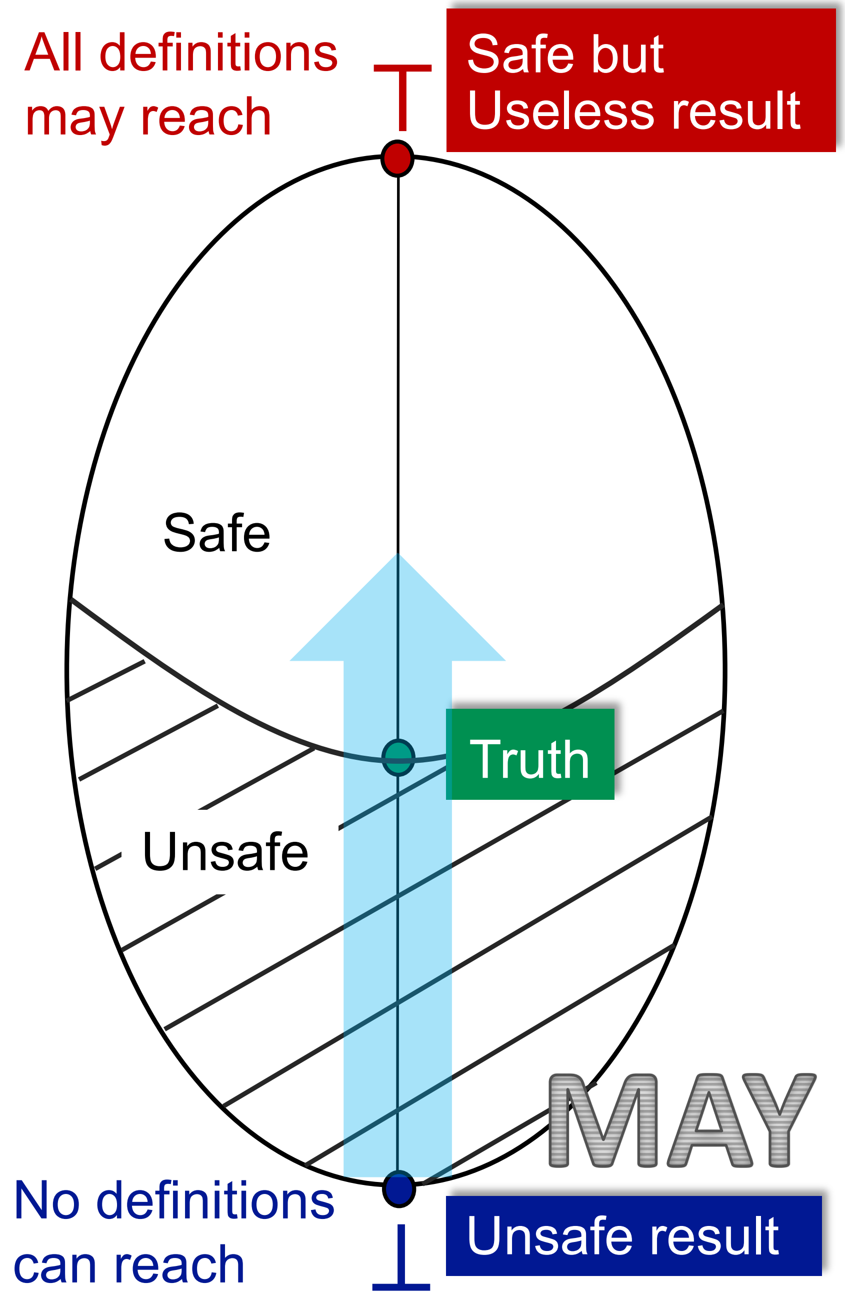
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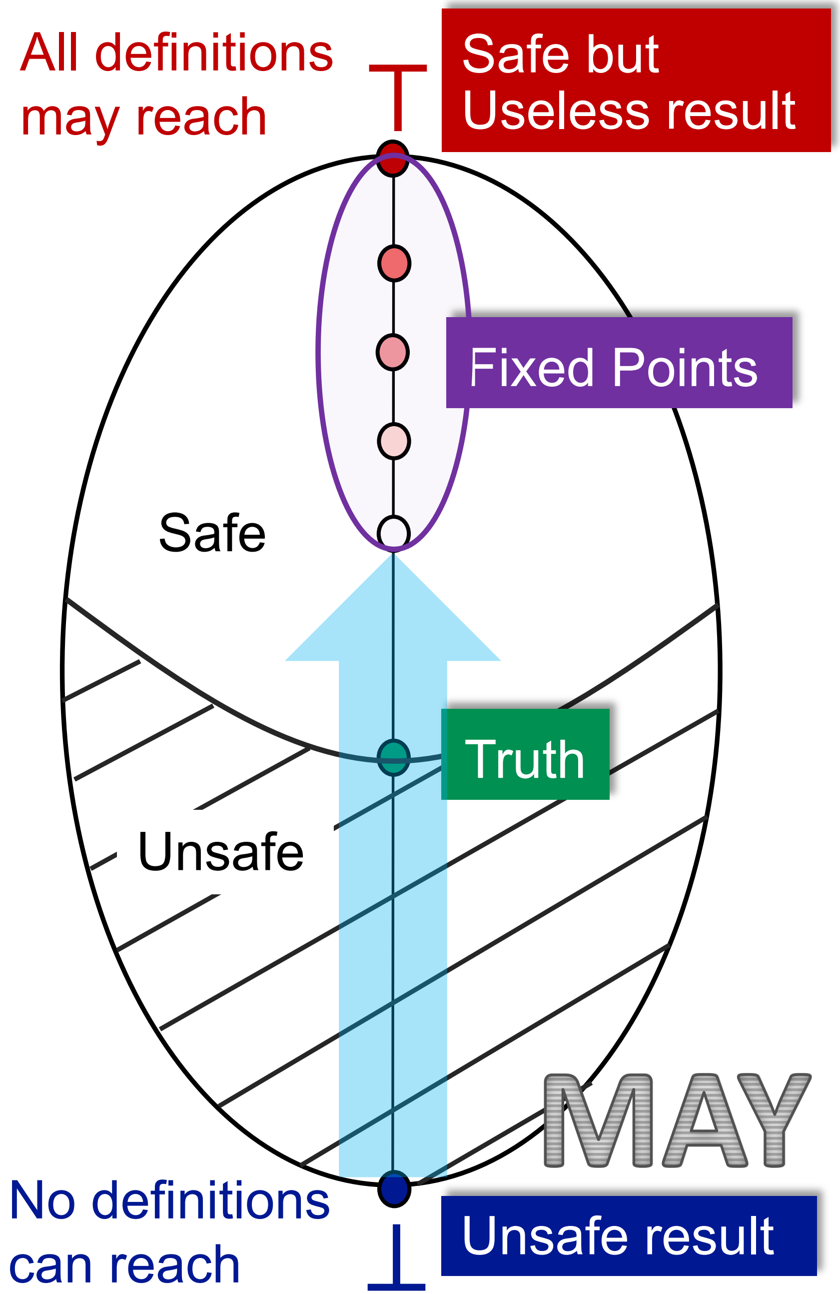
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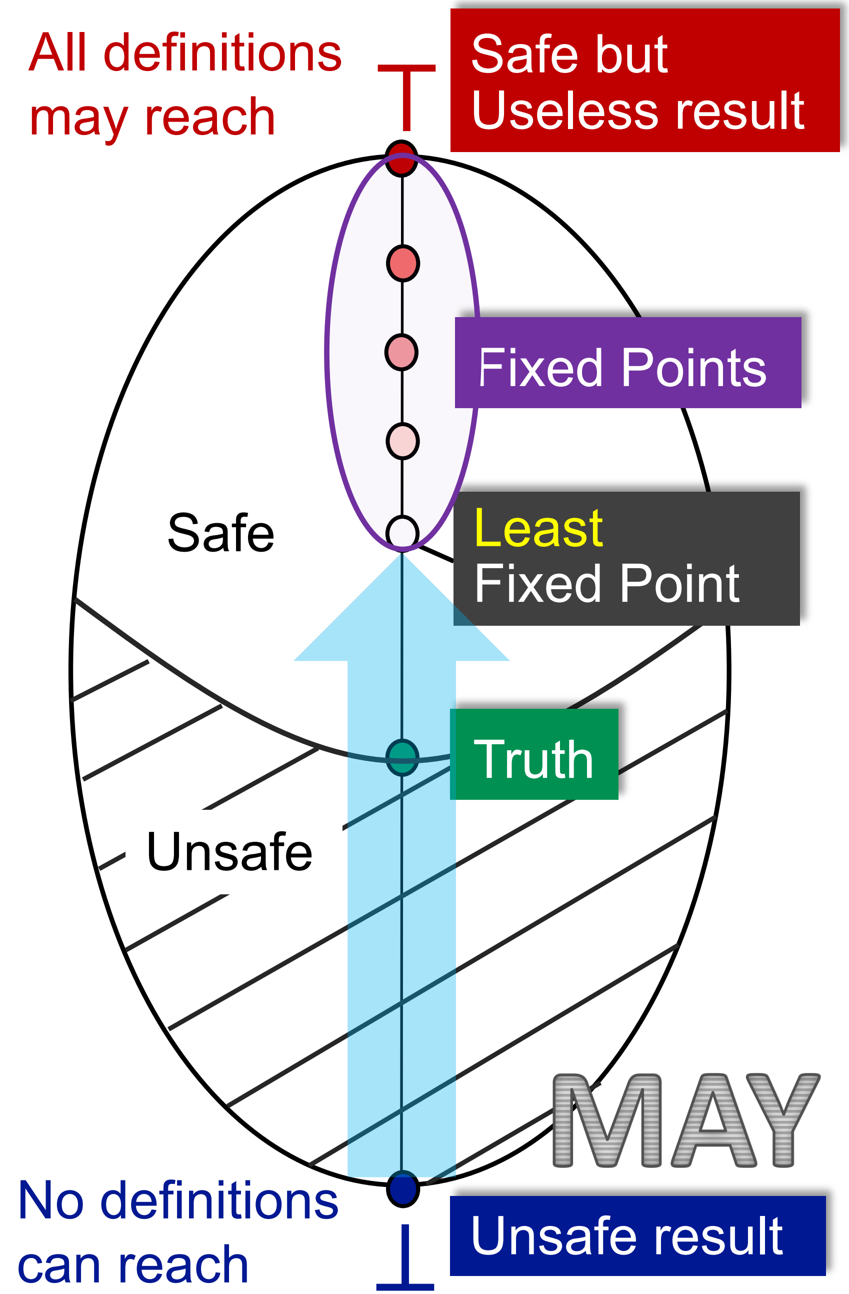
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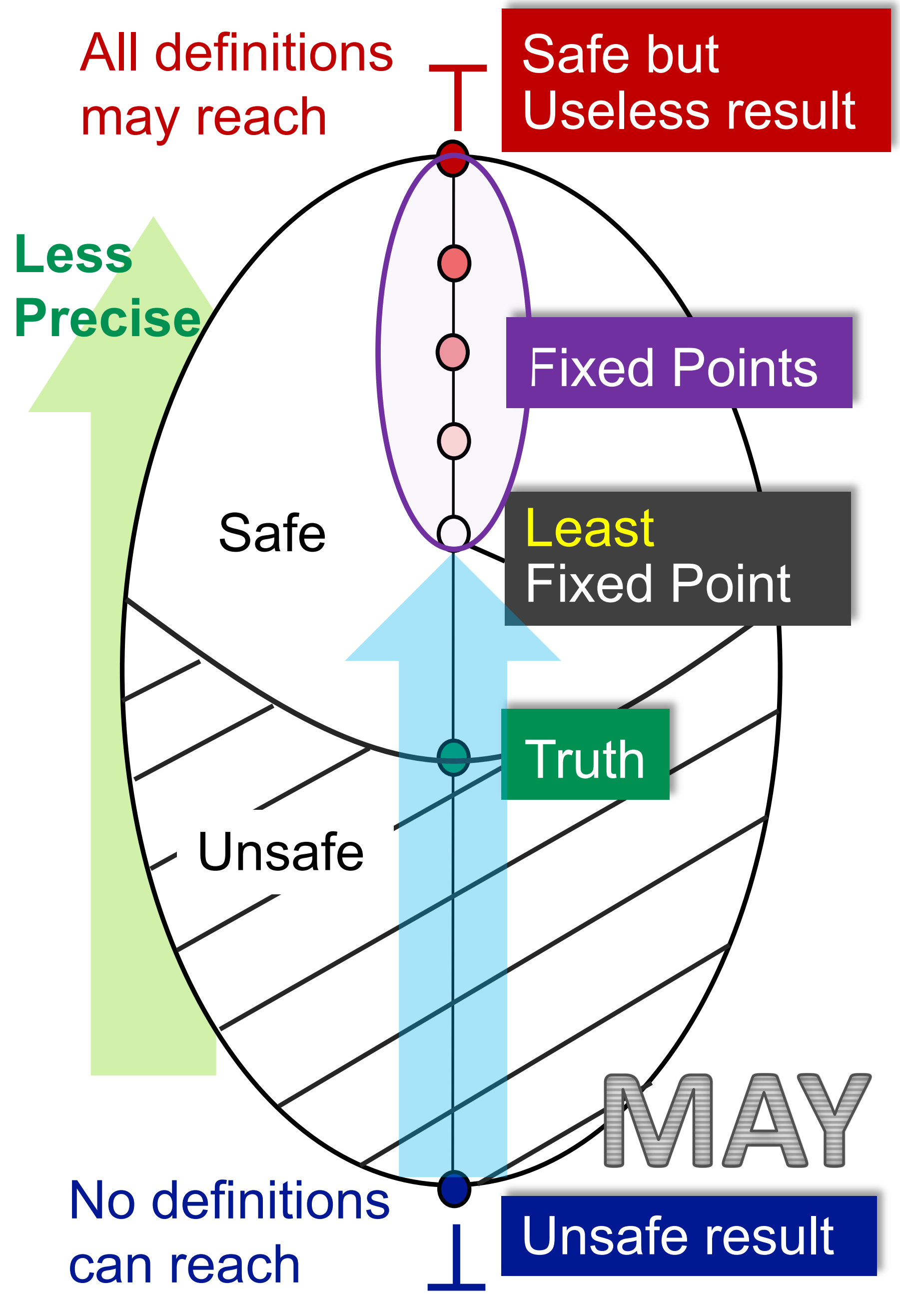
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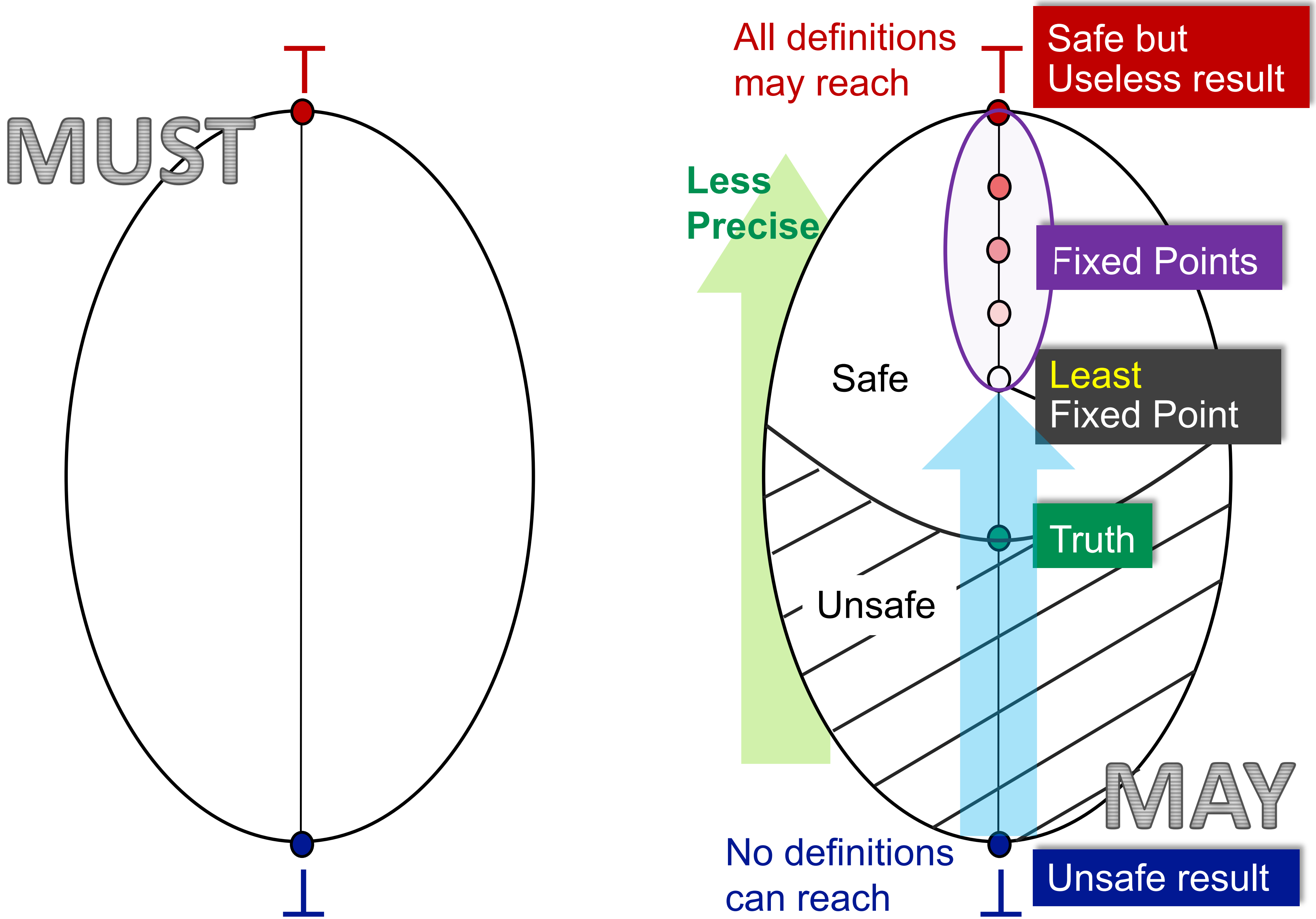
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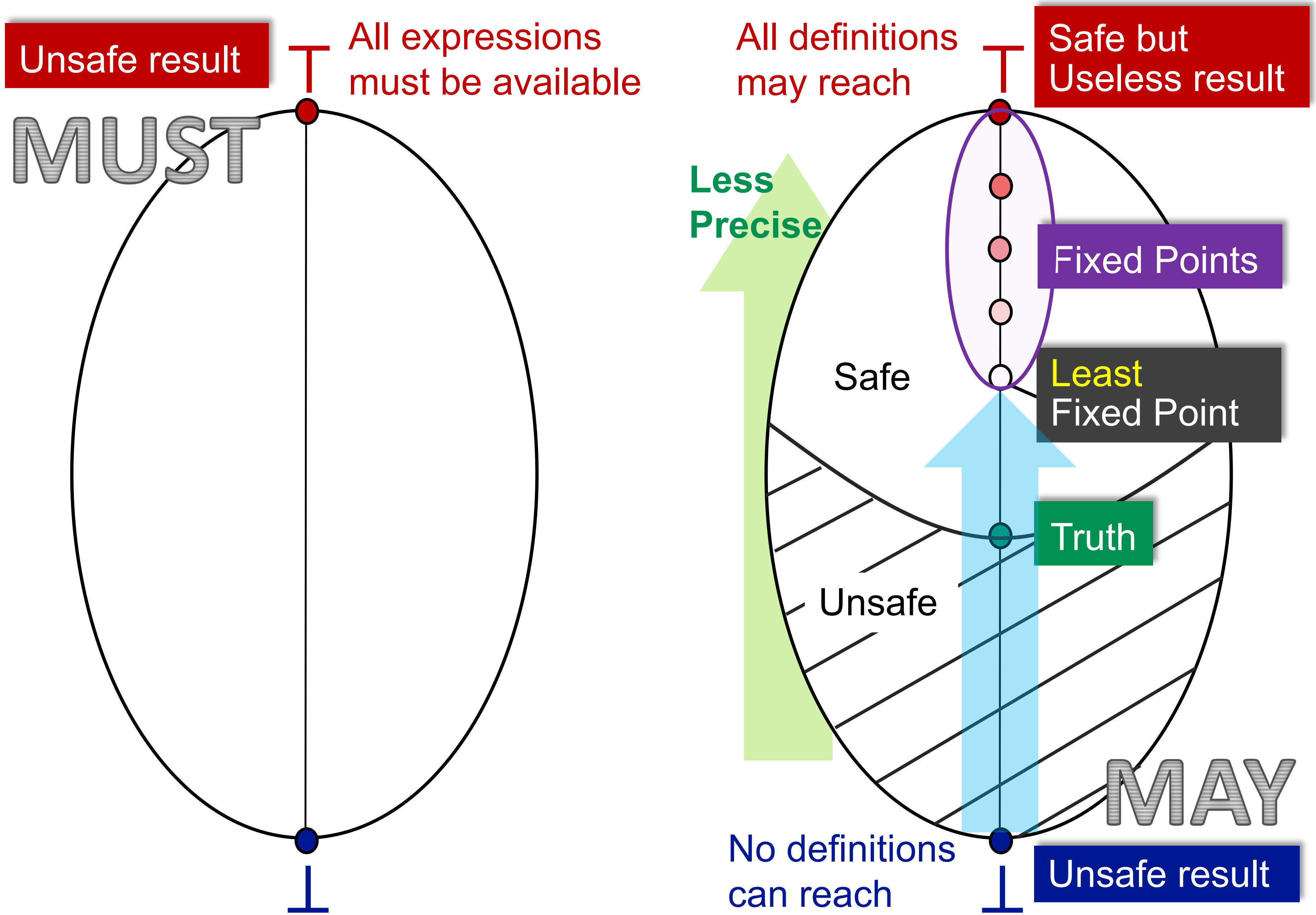
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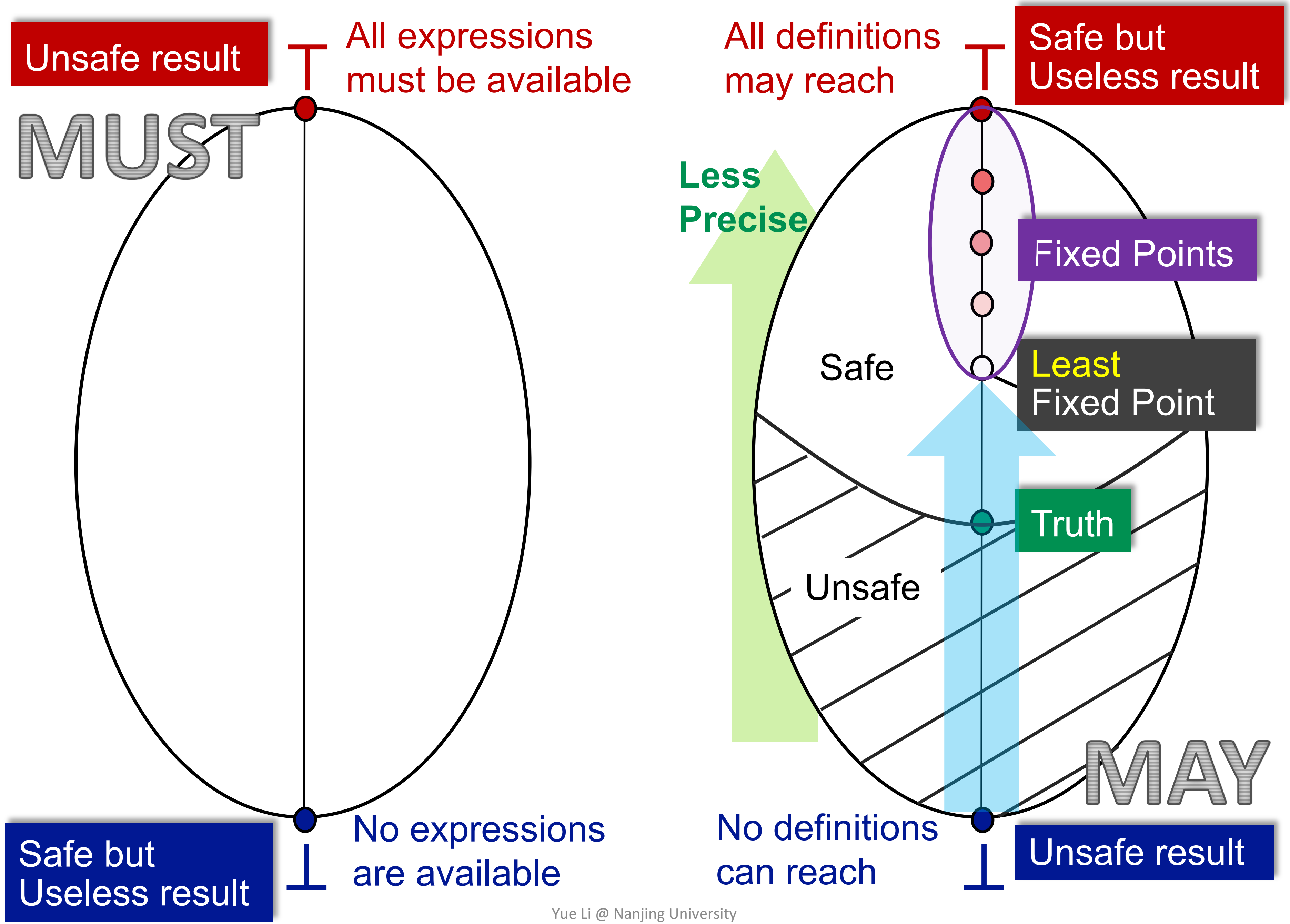

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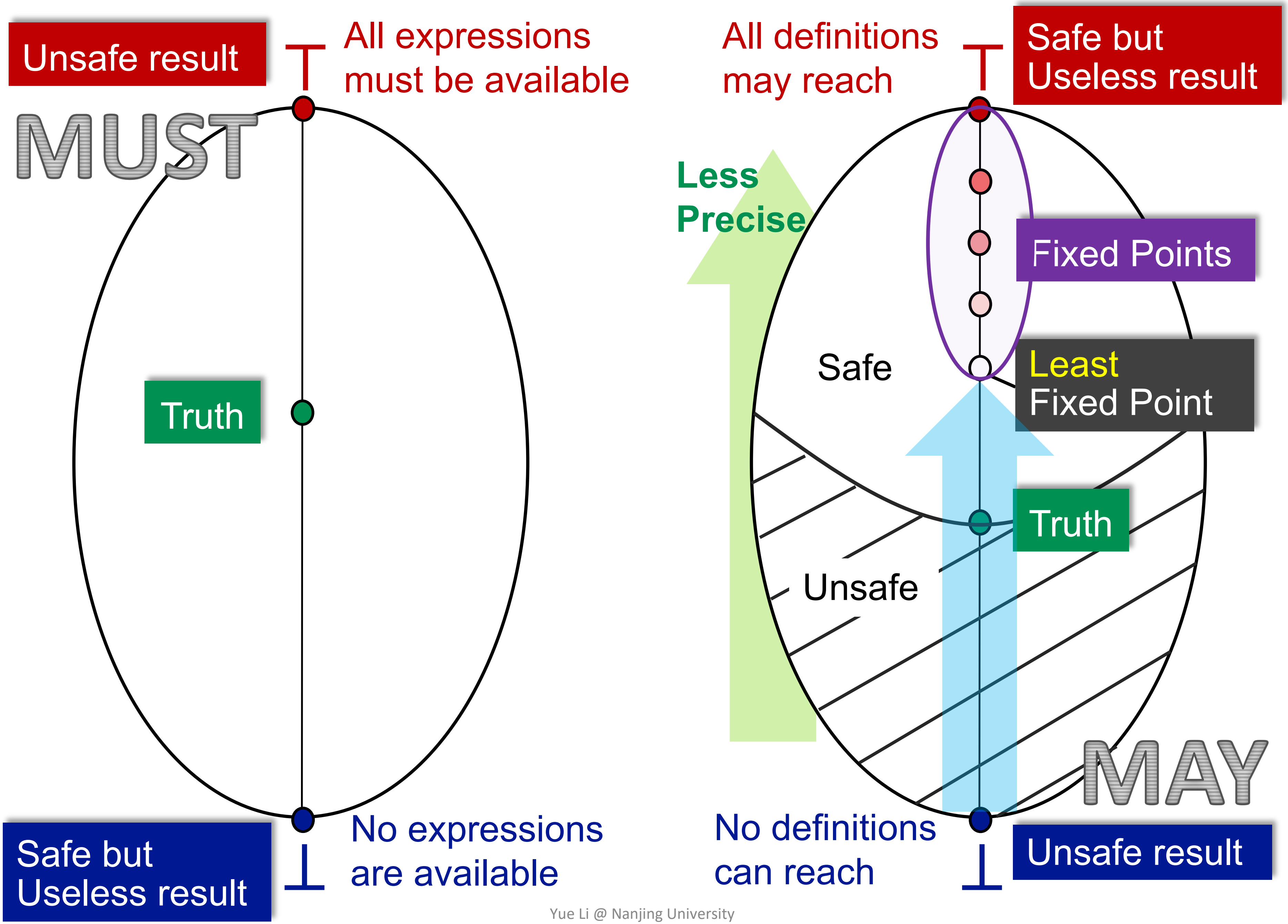

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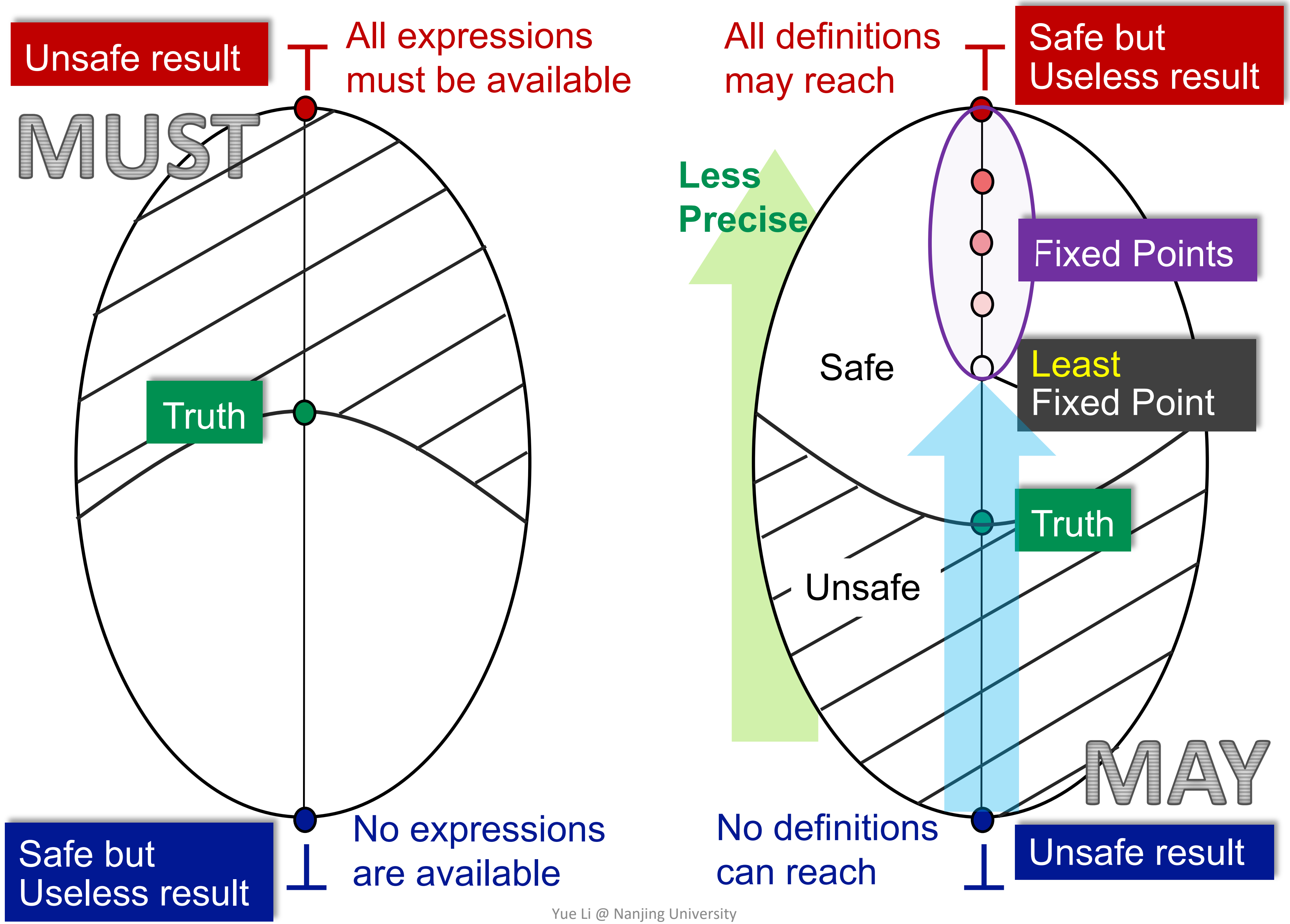

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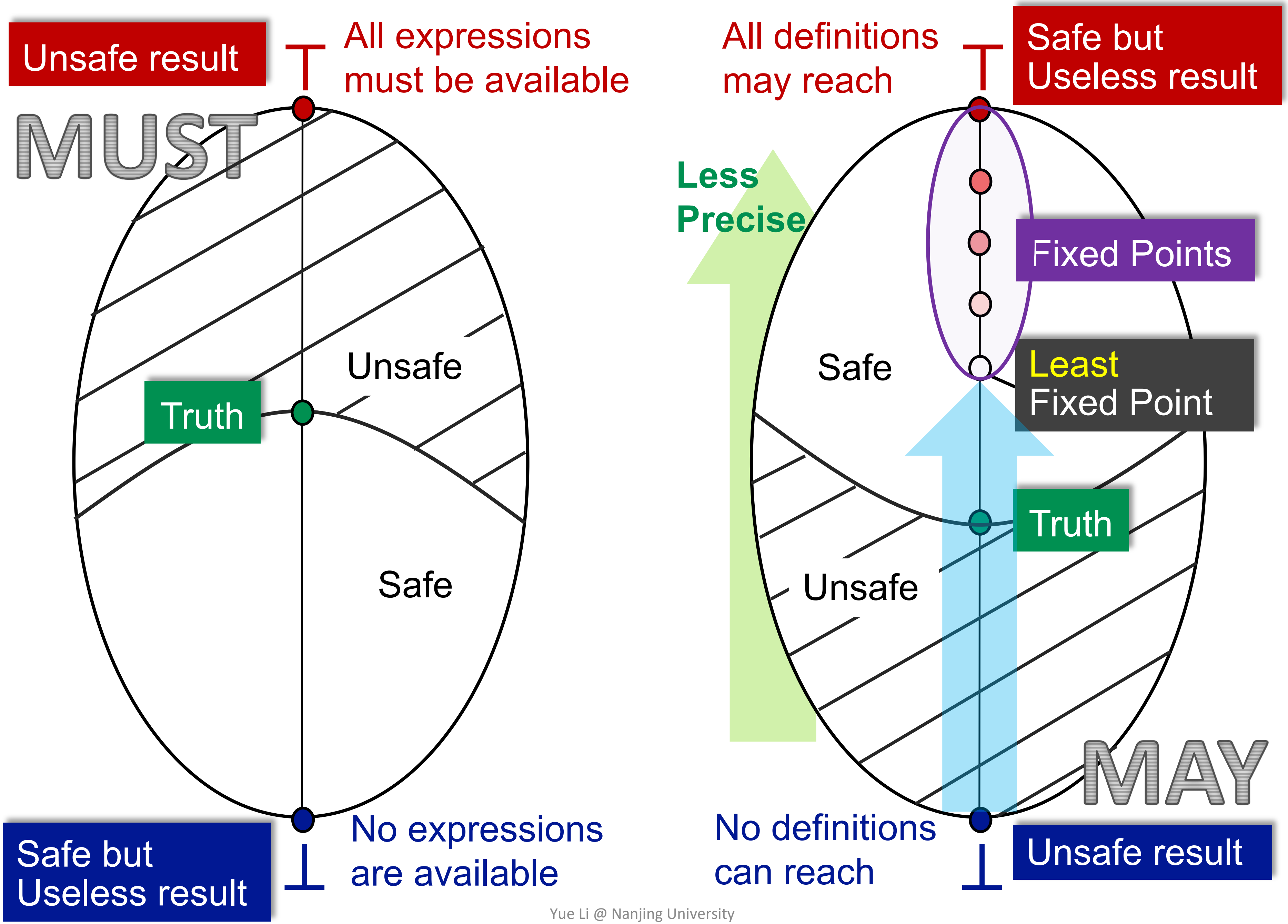
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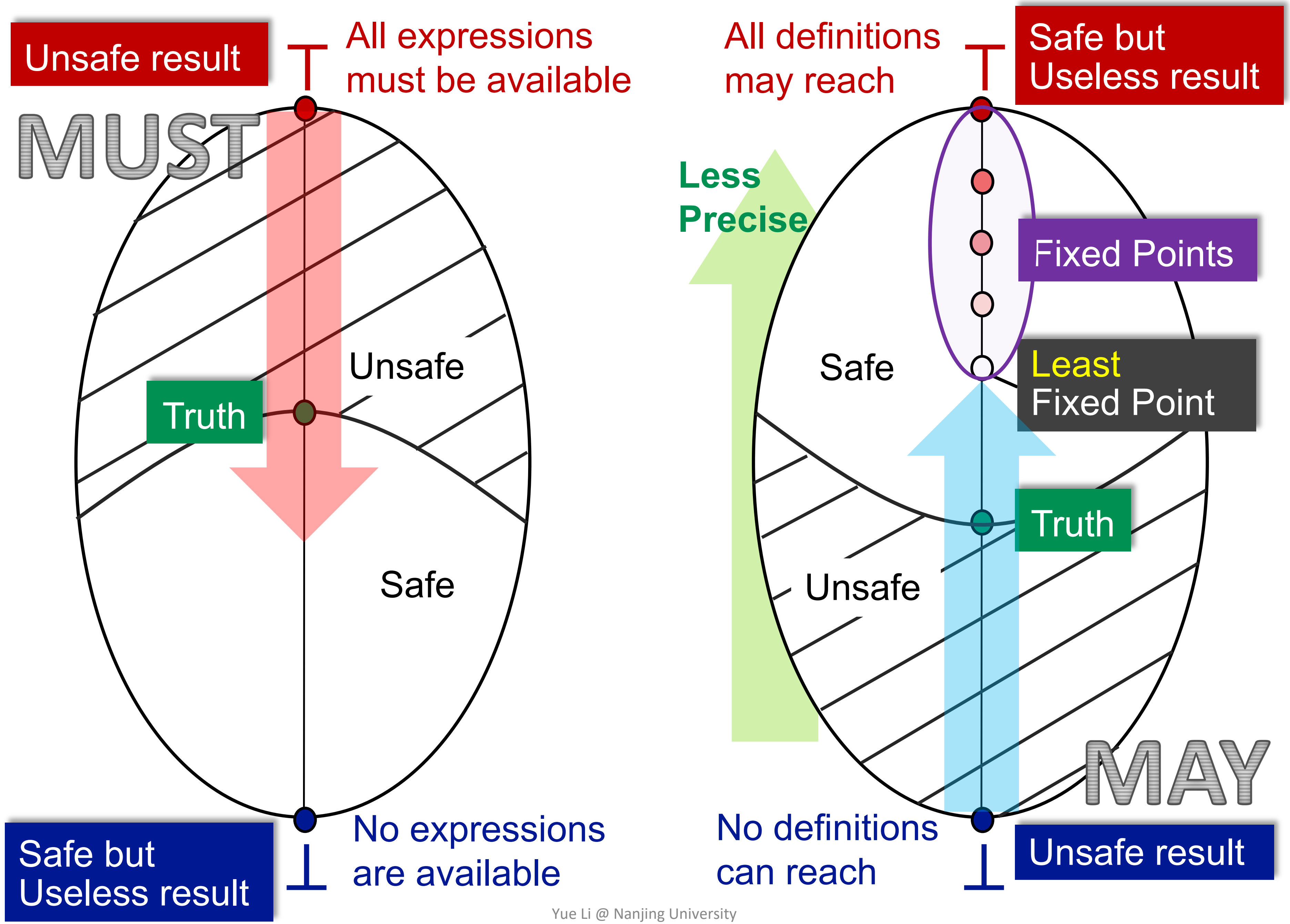
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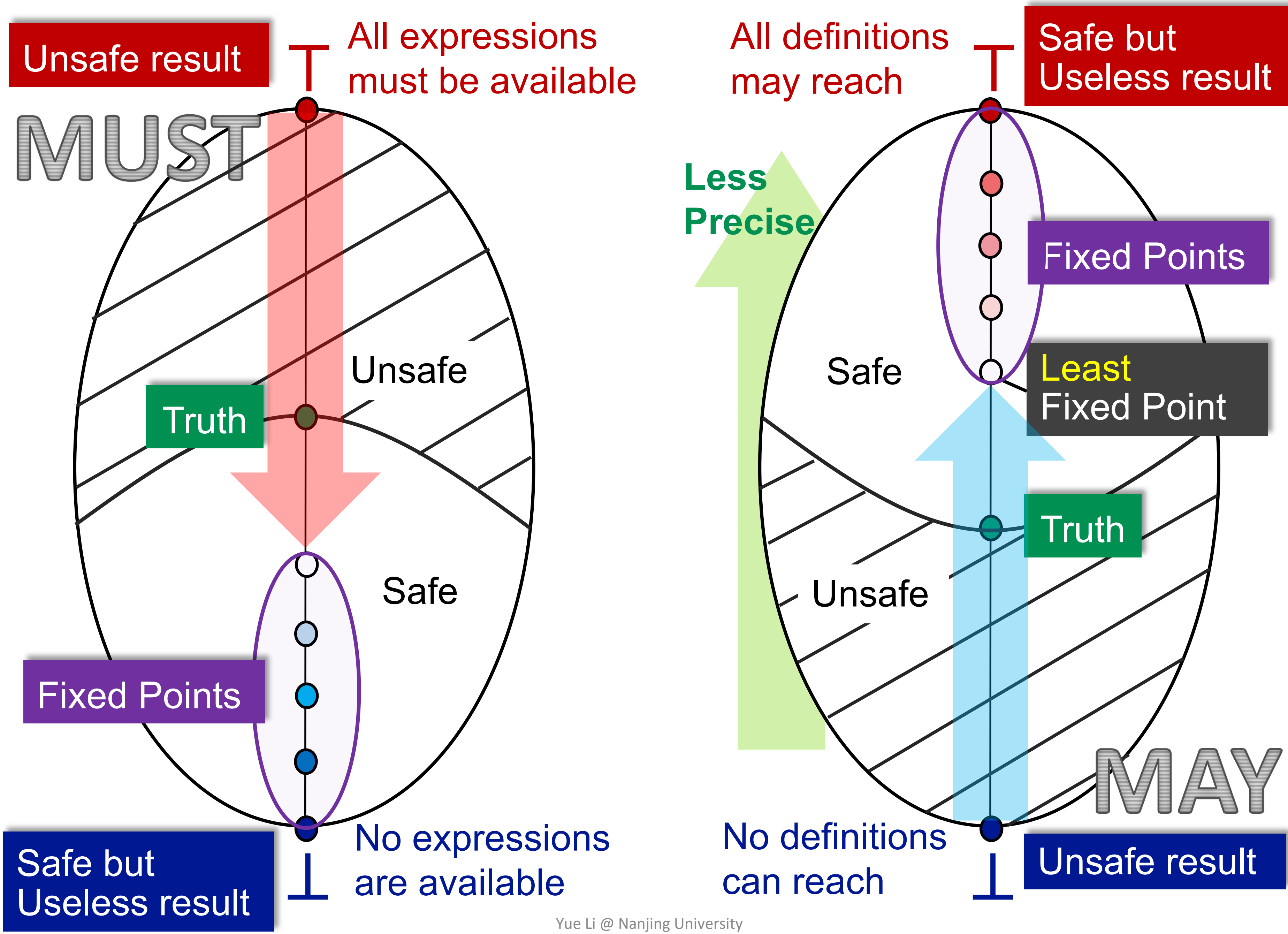
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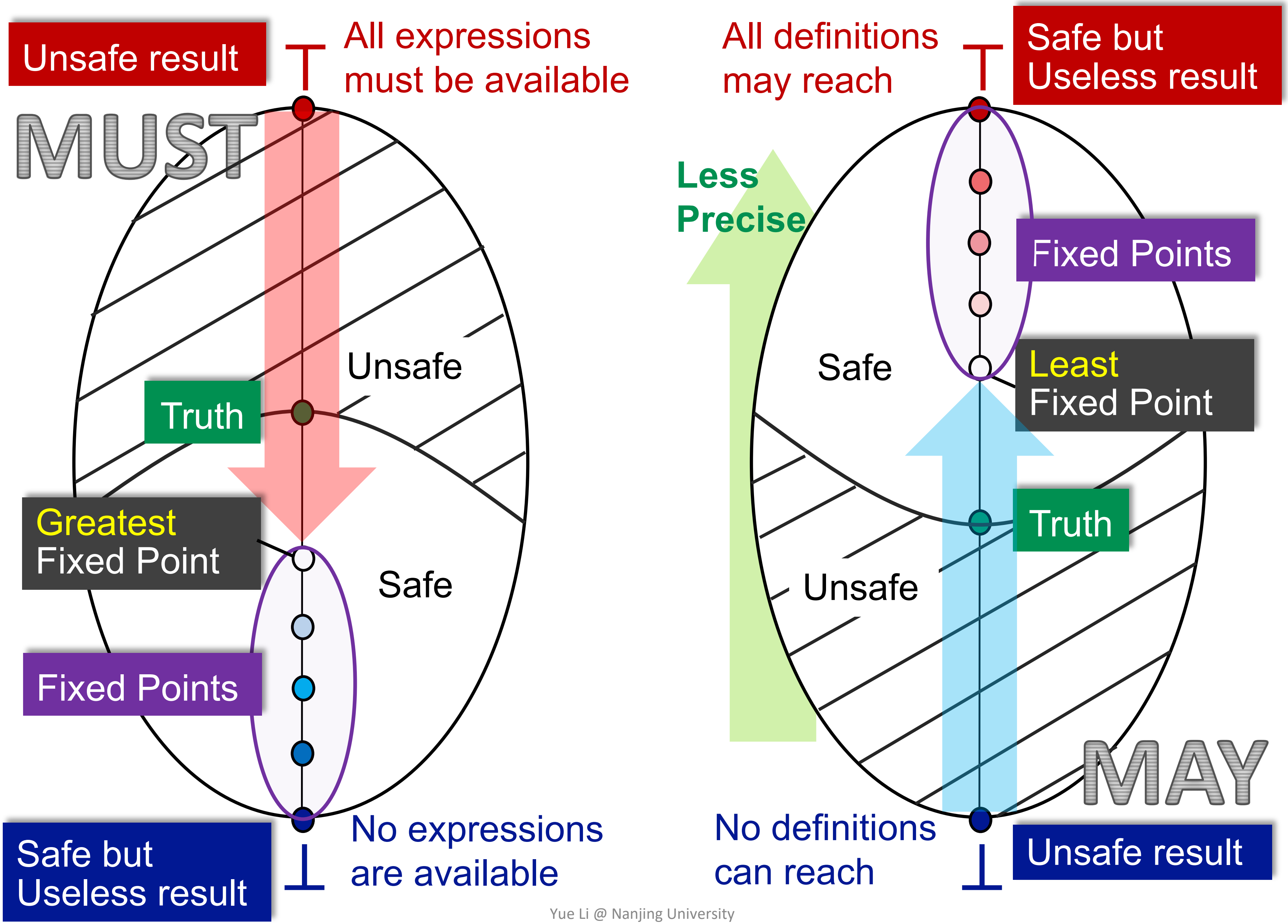

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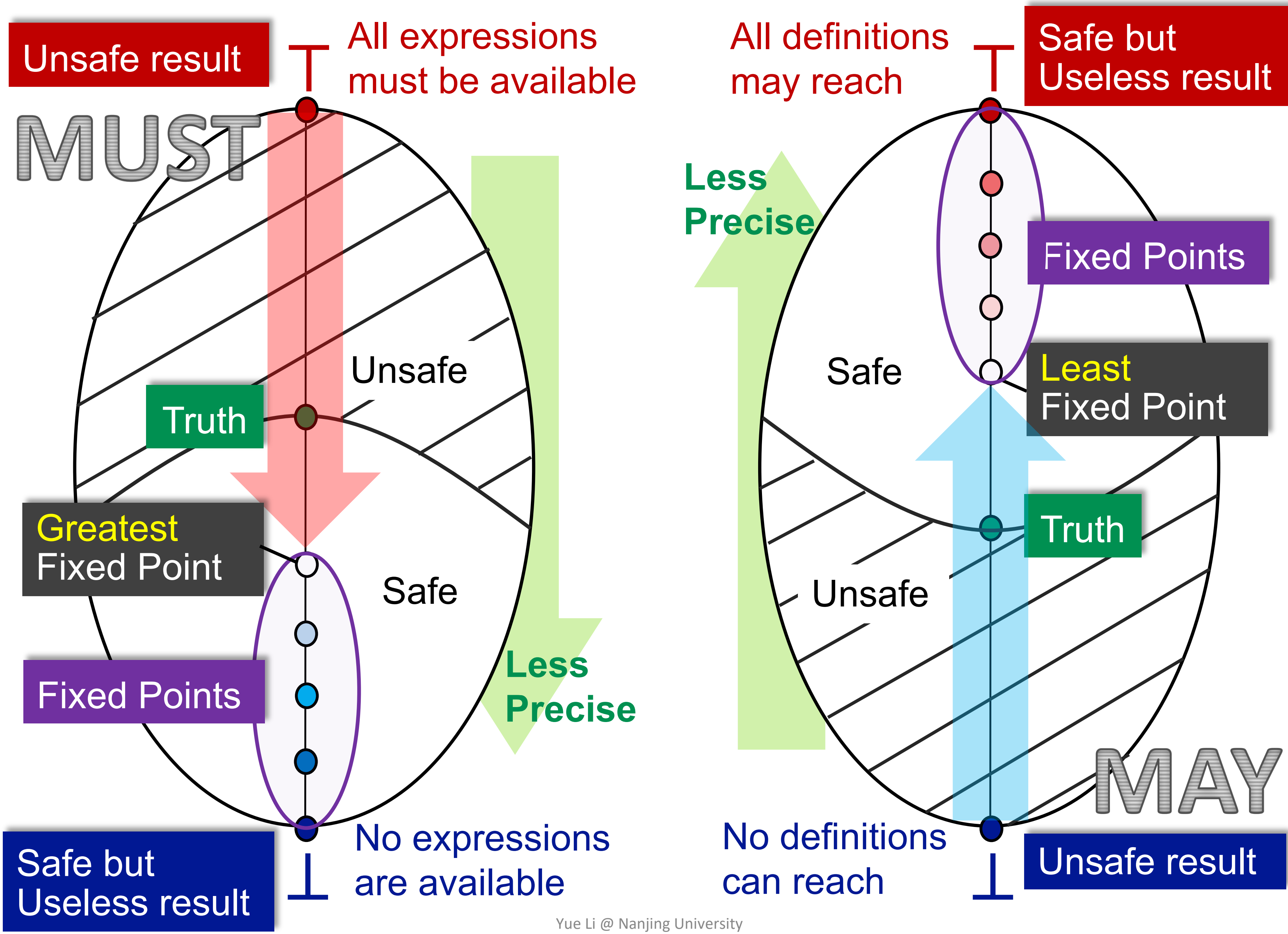

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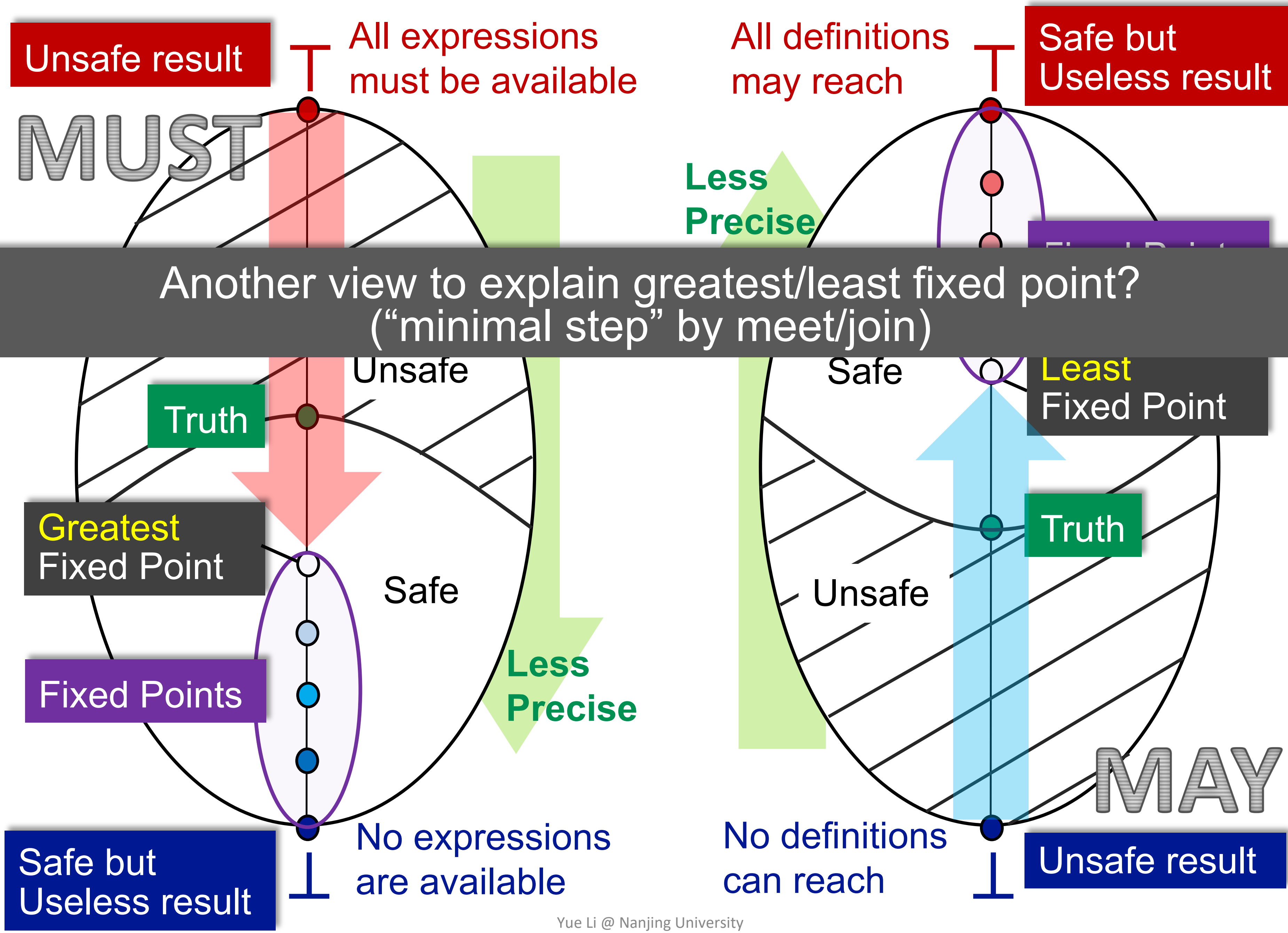
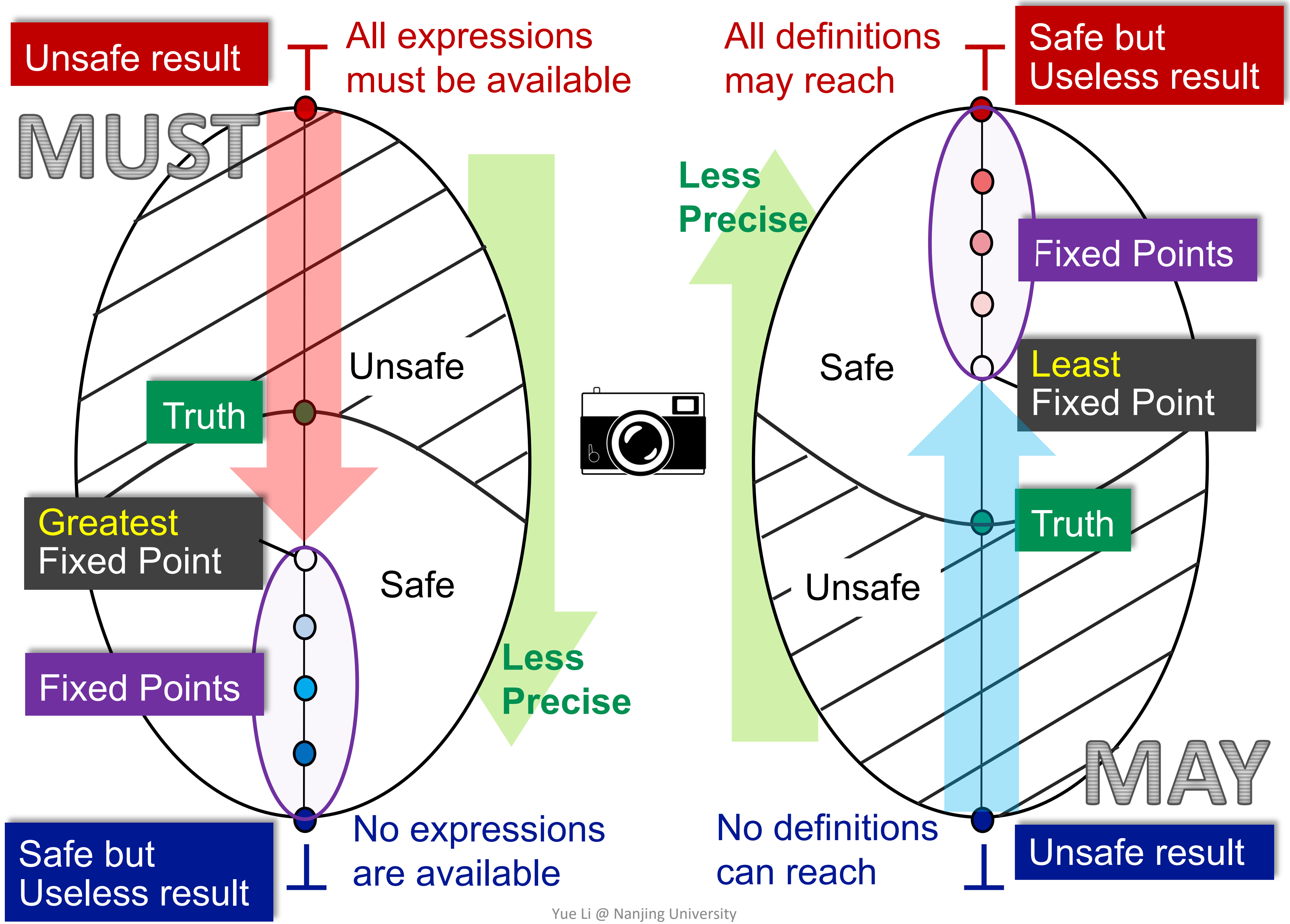


Another view to explain greatest/least fixed point?
(“minimal step” by meet/join)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How Precise Is Our Solution?

- Meet-Over-All-Paths Solution (MOP)

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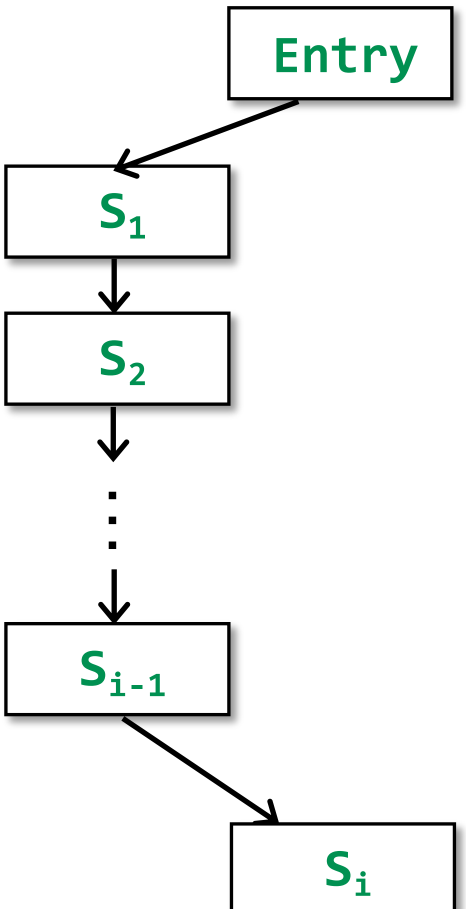
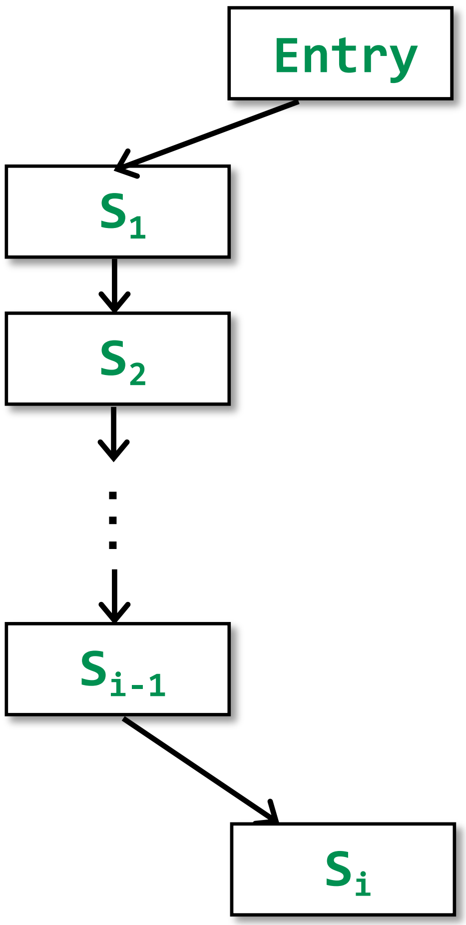


P = Entry → S_1 → S_2 → … → S_i

How Precise Is Our Solution?

- Meet-Over-All-Paths Solution (MOP)

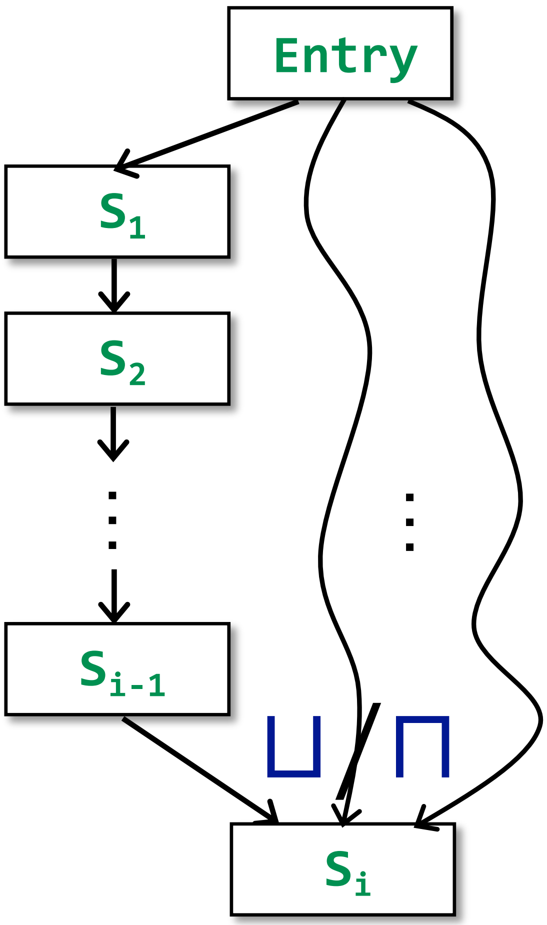


P = Entry → S_1 → S_2 → … → S_i

Transfer function F_P for a path P (from Entry to S_i) is a composition of transfer functions for all statements on that path: f_{S1}, f_{S2}, …, f_{Si-1}

How Precise Is Our Solution?

- Meet-Over-All-Paths Solution (MOP)



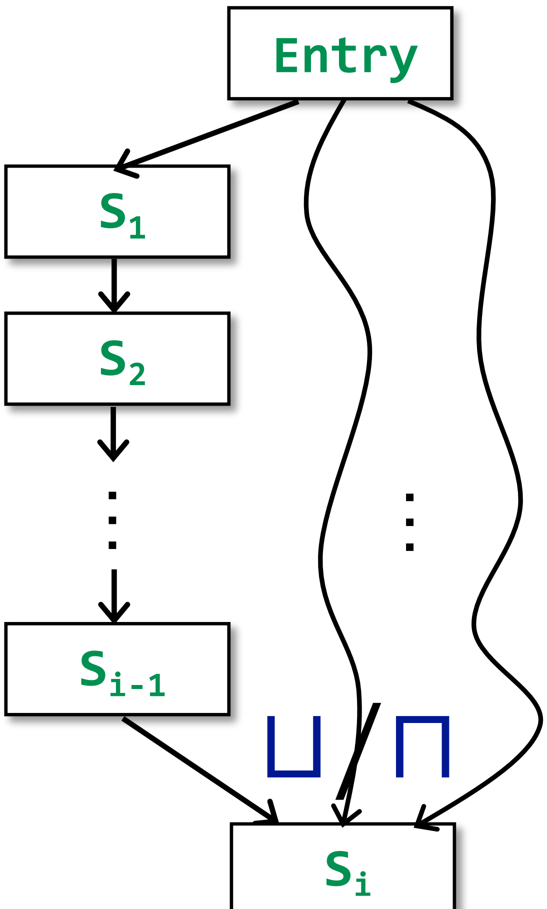
P = `Entry` → S_1 → S_2 → … → S_i

Transfer function F_P for a path P (from Entry to S_i) is a composition of transfer functions for all statements on that path: f_{S1}, f_{S2}, …, f_{Si-1}

$$MOP[S_i] = \underset{\text{A path } P \text{ from Entry to } S_i}{\sqcup / \sqcap} F_P(OUT[\text{Entry}])$$

How Precise Is Our Solution?

- Meet-Over-All-Paths Solution (MOP)



$\mathbf{P} = \mathtt{Entry} \rightarrow \mathtt{S_1} \rightarrow \mathtt{S_2} \rightarrow \ldots \rightarrow \mathtt{S_i}$

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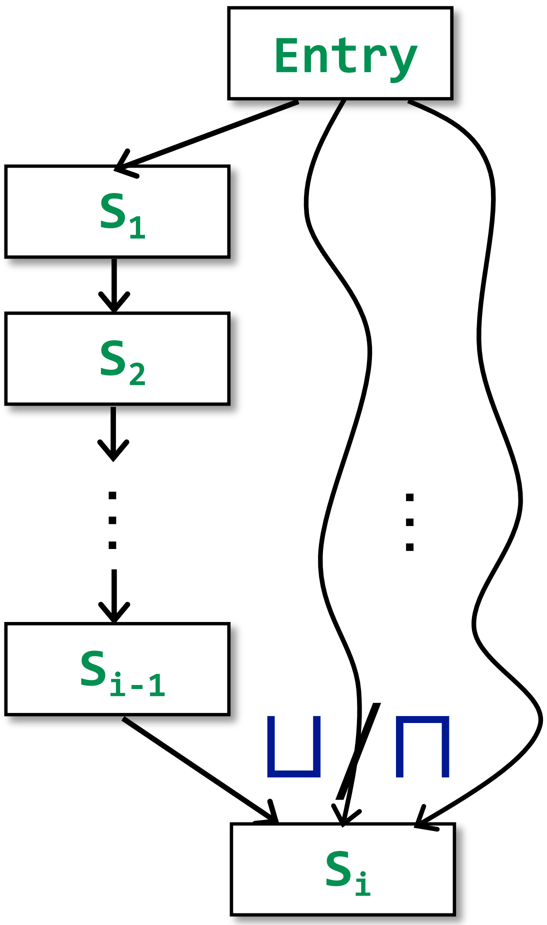
MOP computes the data-flow values at the end of each path and apply join / meet operator to these values to find their lub / glb

How Precise Is Our Solution?

- Meet-Over-All-Paths Solution (MOP)



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Some paths may be not executable ➔ **not fully precise**

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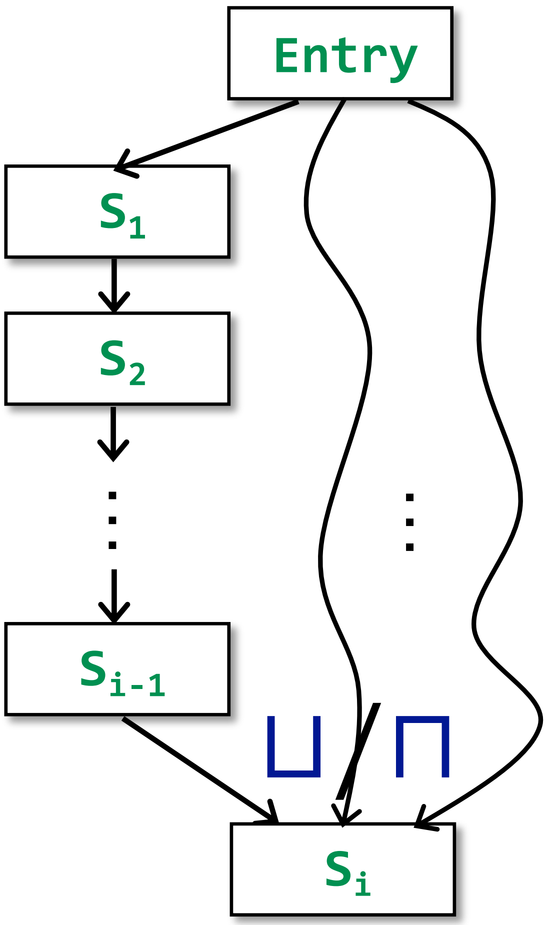

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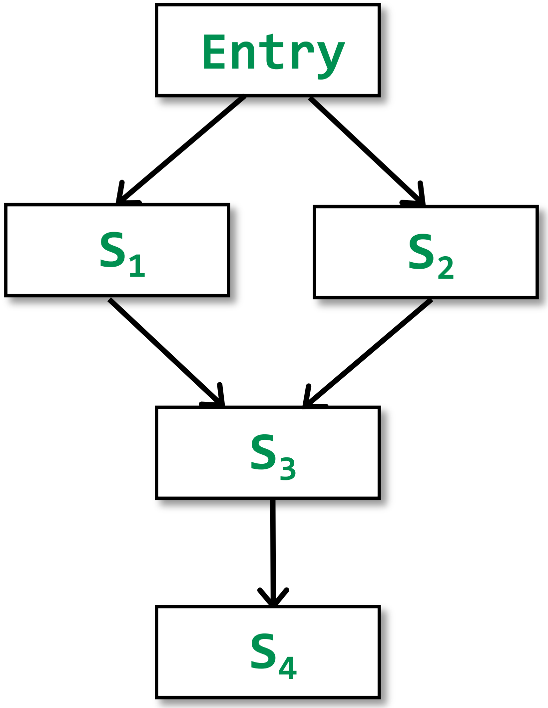
Some paths may be not executable → **not fully precise**

Unbounded, and not enumerable → **impractical**

Ours (Iterative Algorithm) vs. MOP



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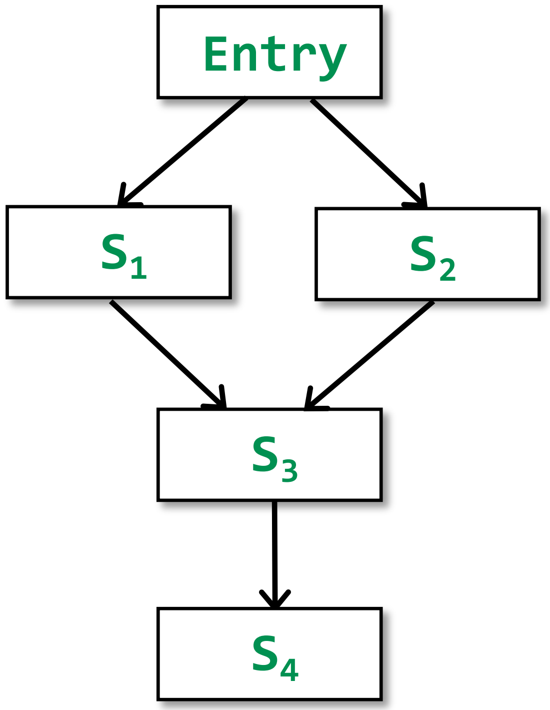
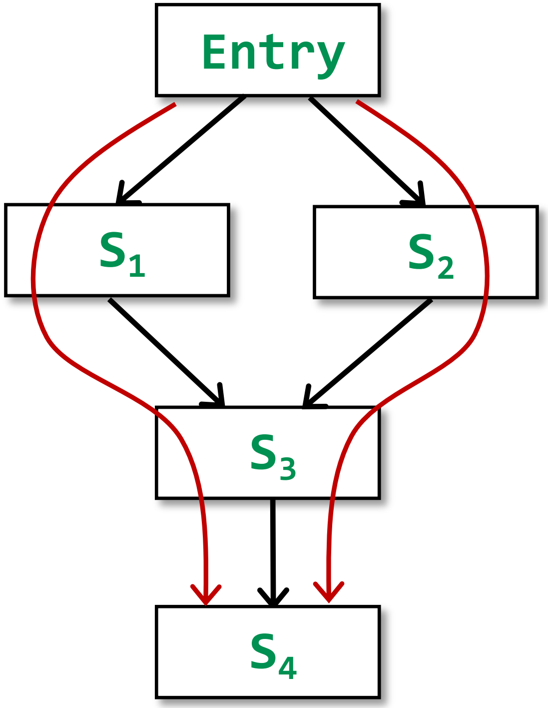


$$\mathrm{IN}[S_4] = f_{S_3}(f_{S_1}(\mathrm{OUT}[\mathtt{Entry}]) \sqcup f_{S_2}(\mathrm{OUT}[\mathtt{Entry}]))$$

Ours (Iterative Algorithm) vs. MOP



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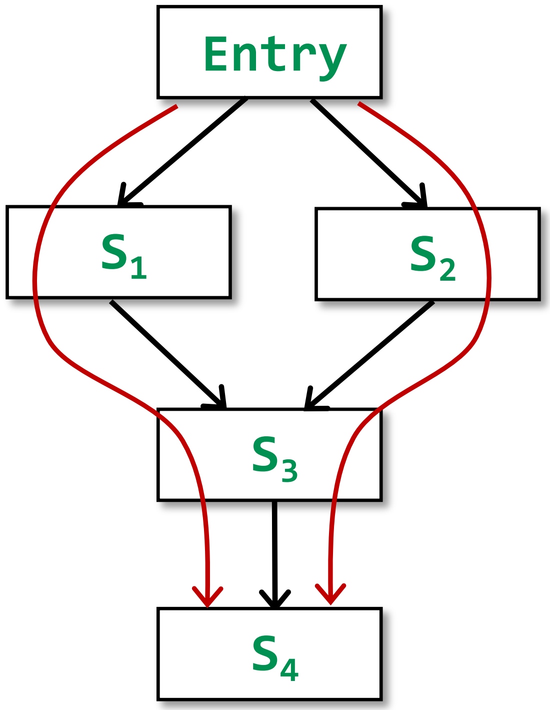
$$\mathrm{MOP}[s_4] = f_{s_3}(f_{s_1}(\mathrm{OUT}[\texttt{Entry}])) \sqcup f_{s_3}(f_{s_2}(\mathrm{OUT}[\texttt{Entry}]))$$

Ours (Iterative Algorithm) vs. MOP



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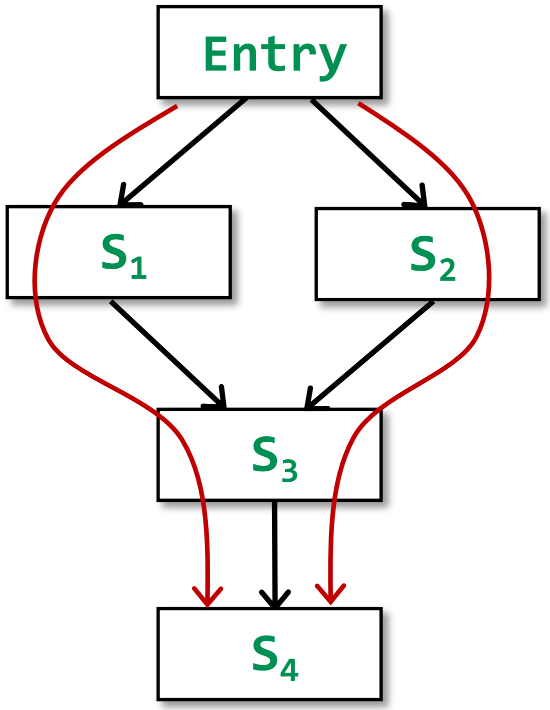

Ours = F(x ⊔ y)

MOP = F(x) ⊔ F(y)

$$\text{IN}[s_4] = f_{s_3}(f_{s_1}(\text{OUT}[\texttt{Entry}]) \sqcup f_{s_2}(\text{OUT}[\texttt{Entry}]))$$

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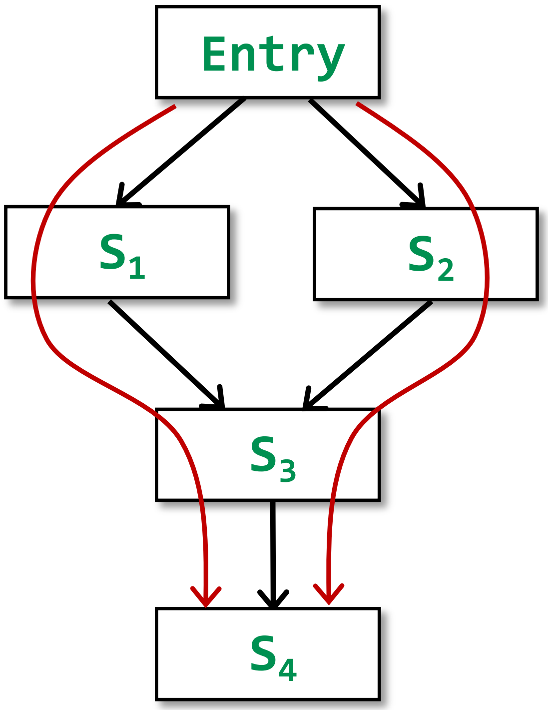
Ours (Iterative Algorithm) vs. MOP



$$\text{Ours} = F(x \sqcup y)$$
$$\text{MOP} = F(x) \sqcup F(y)$$

?

$$\text{IN}[s_4] = f_{s_3}(f_{s_1}(\text{OUT}[\texttt{Entry}]) \sqcup f_{s_2}(\text{OUT}[\texttt{Entry}]))$$

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Ours (Iterative Algorithm) vs. MOP

Ours = $F(x \sqcup y)$

MOP = $F(x) \sqcup F(y)$

Ours (Iterative Algorithm) vs. MOP

Ours = $F(x \sqcup y)$

MOP = $F(x) \sqcup F(y)$

By definition of lub $\sqcup$, we have

$$x \sqsubseteq x \sqcup y \text{ and } y \sqsubseteq x \sqcup y$$

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$$F(x) \sqsubseteq F(x \sqcup y) \text{ and } F(y) \sqsubseteq F(x \sqcup y)$$

That means $F(x \sqcup y)$ is an upper bound of $F(x)$ and $F(y)$

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$$\text{MOP} \sqsubseteq \text{Ours}$$

Ours (Iterative Algorithm) vs. MOP

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$$\text{MOP} \sqsubseteq \text{Ours}$$

(Ours is less precise than MOP)

Ours (Iterative Algorithm) vs. MOP

Ours = $F(x \sqcup y)$

MOP = $F(x) \sqcup F(y)$

By definition of lub $\sqcup$, we have

$$x \sqsubseteq x \sqcup y \text{ and } y \sqsubseteq x \sqcup y$$

As transfer function F is **monotonic**, we have

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That means $F(x \sqcup y)$ is an upper bound of $F(x)$ and $F(y)$

As $F(x) \sqcup F(y)$ is the lub of $F(x)$ and $F(y)$, we have

$$F(x) \sqcup F(y) \sqsubseteq F(x \sqcup y)$$

$$\text{MOP} \sqsubseteq \text{Ours}$$

(Ours is less precise than MOP)

When F is **distributive**, i.e.,

$$F(x \sqcup y) = F(x) \sqcup F(y)$$

Ours (Iterative Algorithm) vs. MOP

Ours = $F(x \sqcup y)$

MOP = $F(x) \sqcup F(y)$

By definition of lub $\sqcup$, we have

$$x \sqsubseteq x \sqcup y \text{ and } y \sqsubseteq x \sqcup y$$

As transfer function F is **monotonic**, we have

$$F(x) \sqsubseteq F(x \sqcup y) \text{ and } F(y) \sqsubseteq F(x \sqcup y)$$

That means $F(x \sqcup y)$ is an upper bound of $F(x)$ and $F(y)$

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$$\text{MOP} \sqsubseteq \text{Ours}$$

(Ours is less precise than MOP)

When F is **distributive**, i.e.,

$$F(x \sqcup y) = F(x) \sqcup F(y)$$

$$\text{MOP} = \text{Ours}$$

Ours (Iterative Algorithm) vs. MOP

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F ... $F(x \sqcup y)$

M...

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Bit-vector or Gen/Kill problems (set union /intersection for join/meet) are distributive

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Bit-vector or Gen/Kill prob...
/intersection for join/m... ...set union
...re distributive

But some analyses are not distributive

When F is **distributiv...**

$$F(x \sqcup y) = F(x) \sqcup F(y)$$

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Given a variable x at program point p, determine whether x is guaranteed to hold a constant value at p.

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A data flow analysis framework (D, L, F) consists of:

- **D**: a direction of data flow: forwards or backwards
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Constant Propagation – Lattice

- Domain of the values V

- Meet Operator ⊓

Constant Propagation – Lattice

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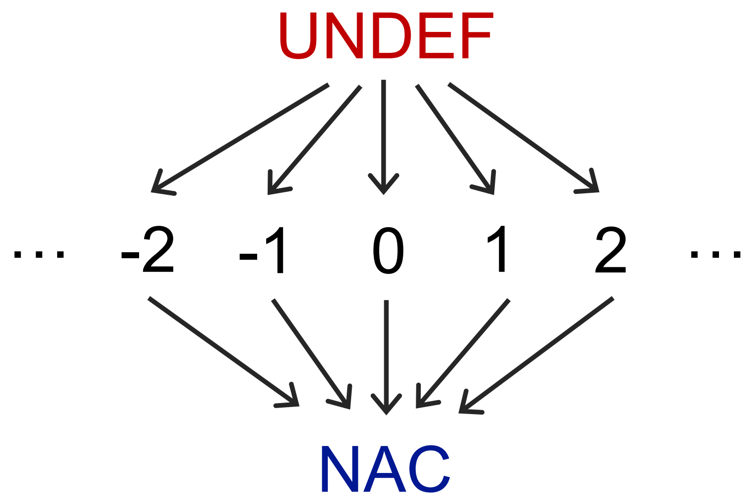

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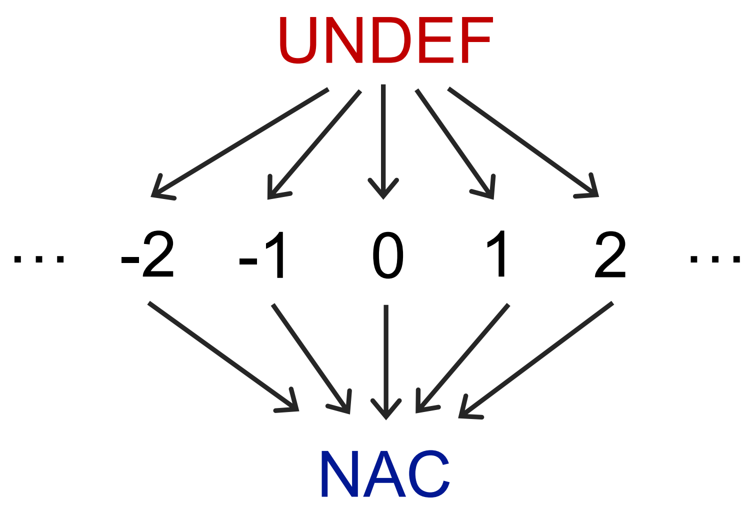
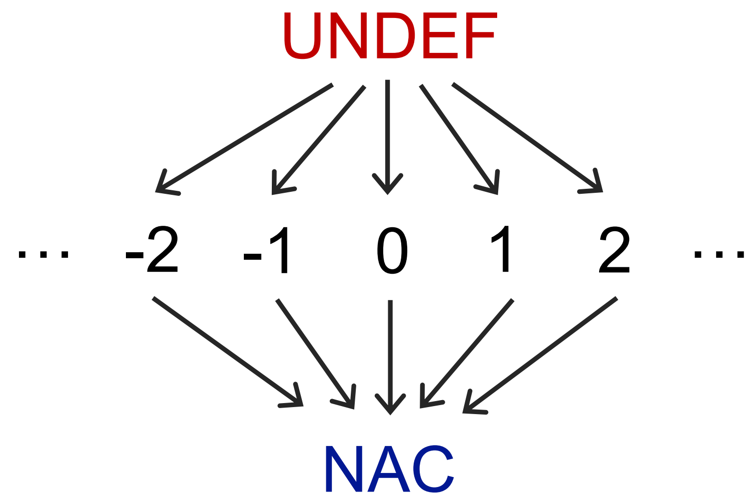


- Meet Operator ⊓

NAC ⊓ v = NAC

Constant Propagation – Lattice

- Domain of the values V



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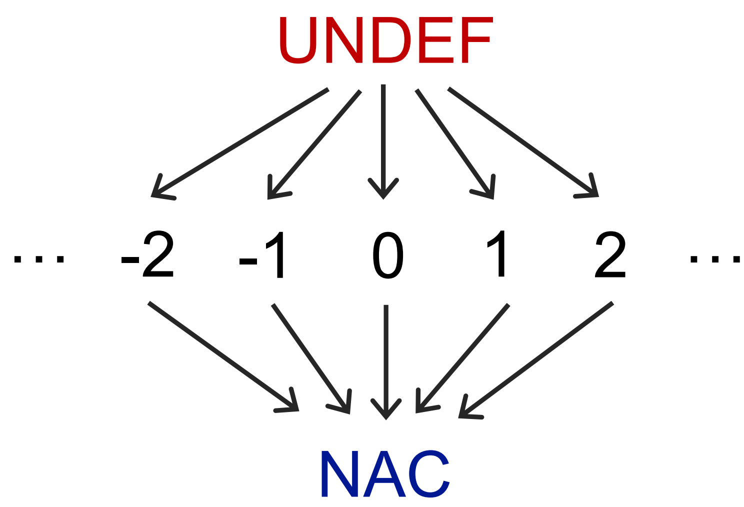

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Uninitialized variables are not the focus in our constant propagation analysis

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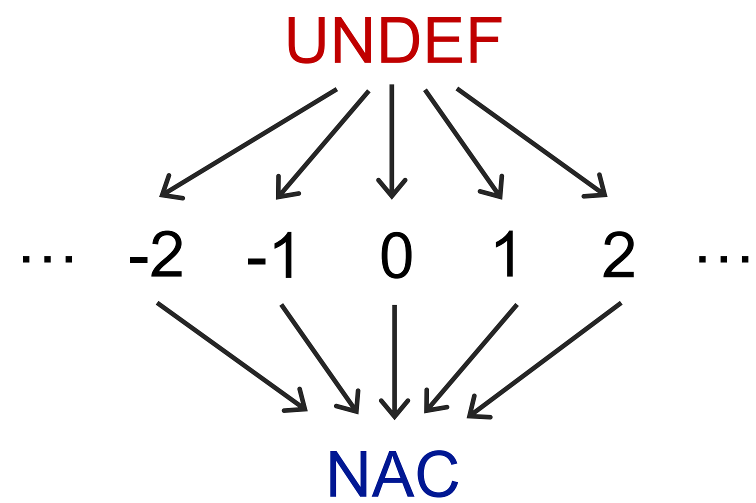

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c ⊓ v = ?

Constant Propagation – Lattice

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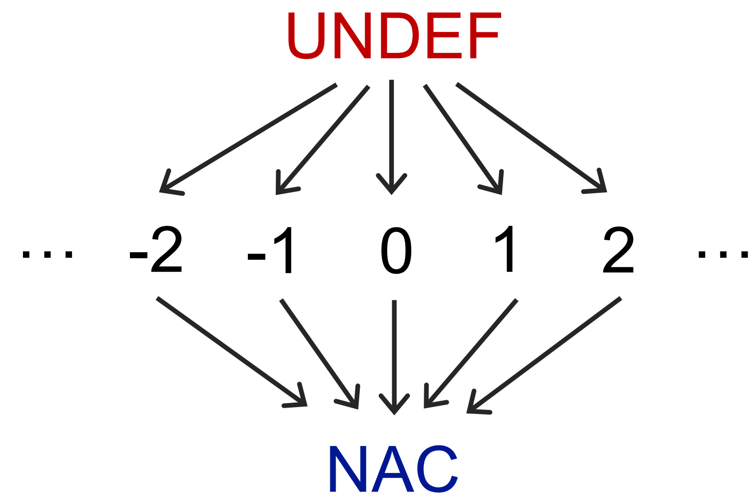

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Uninitialized variables are not the focus in our constant propagation analysis

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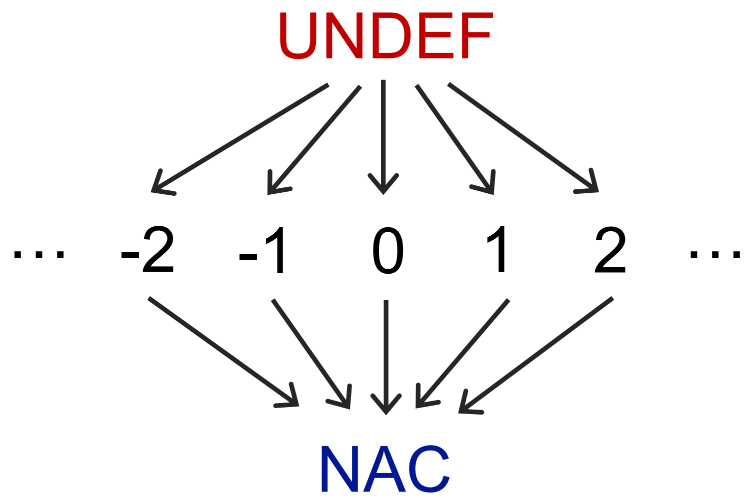
- $c \sqcap c = c$
- $c_1 \sqcap c_2$ = NAC

Constant Propagation – Lattice

- Domain of the values V



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Uninitialized variables are not the focus in our constant propagation analysis

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- $c \sqcap c = c$
- $c_1 \sqcap c_2$ = NAC

At each path confluence PC, we should apply “meet” for all variables in the incoming data-flow values at that PC

Constant Propagation – Transfer Function

Given a statement s: x = …, we define its transfer function F as

$$\boxed{F: OUT[s] = gen \cup (IN[s] - \{(x, \_)\})}$$

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- s: x = y;

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$$f(y,z) = \begin{cases} val(y)\ op\ val(z) & \text{// if val(y) and val(z) are constants} \\ NAC & \text{// if val(y) or val(z) is NAC} \\ UNDEF & \text{// otherwise} \end{cases}$$

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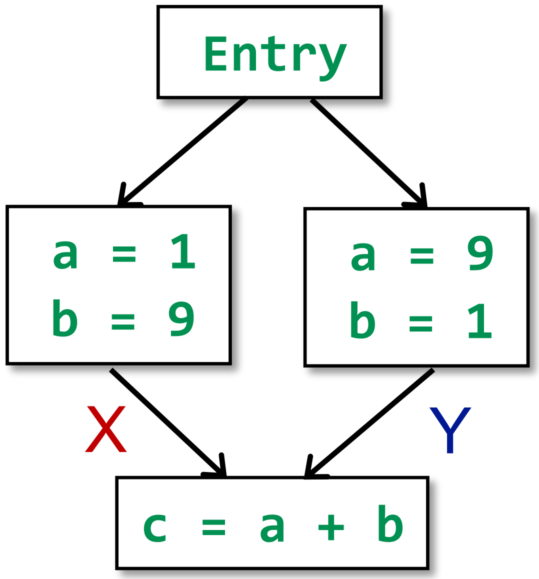
(if s is not an assignment statement, F is the identity function)

Constant Propagation – Nondistributivity



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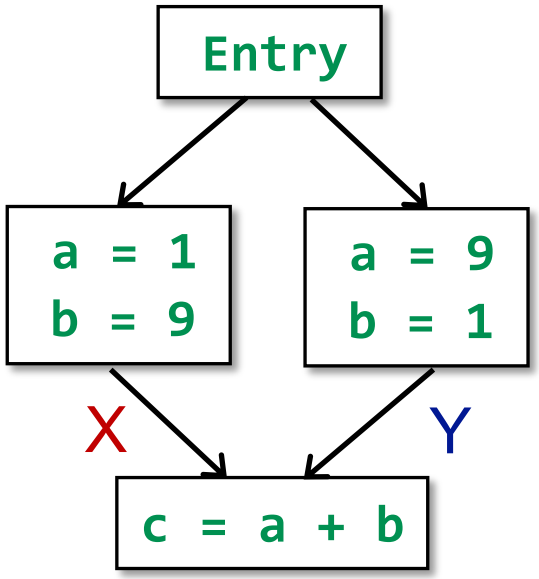

$F(X \sqcap Y) =$

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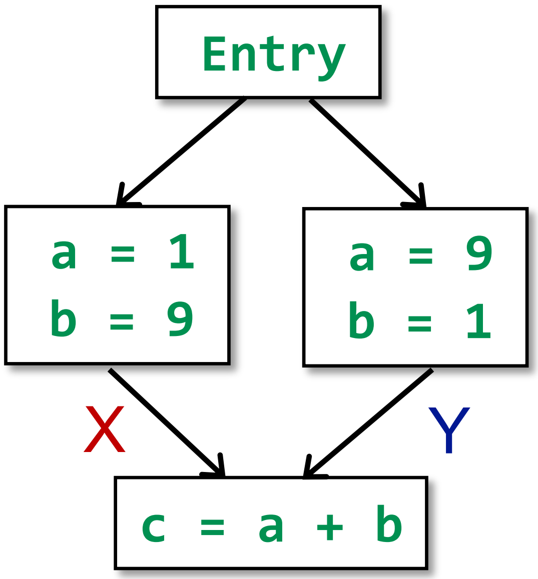

$$F(X \sqcap Y) = \{(a, NAC), (b, NAC), (c, NAC)\}$$
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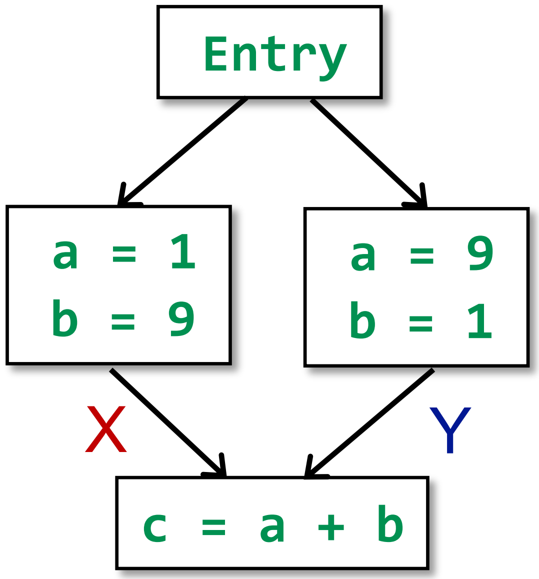
Constant Propagation – Nondistributivity



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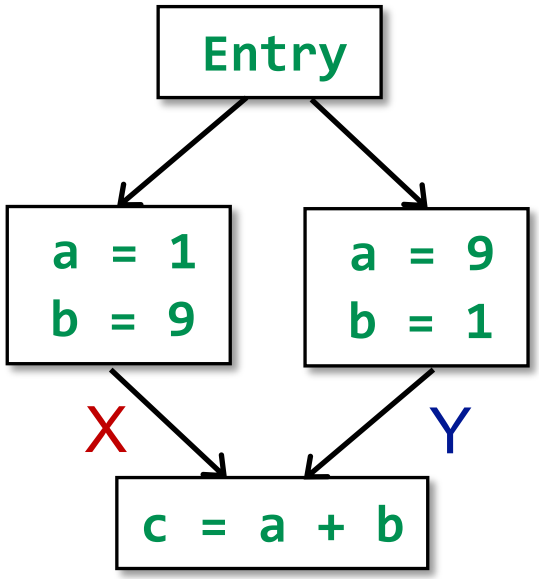

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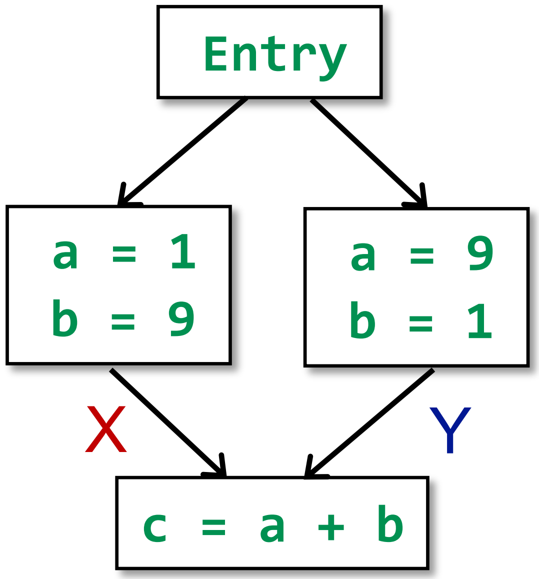

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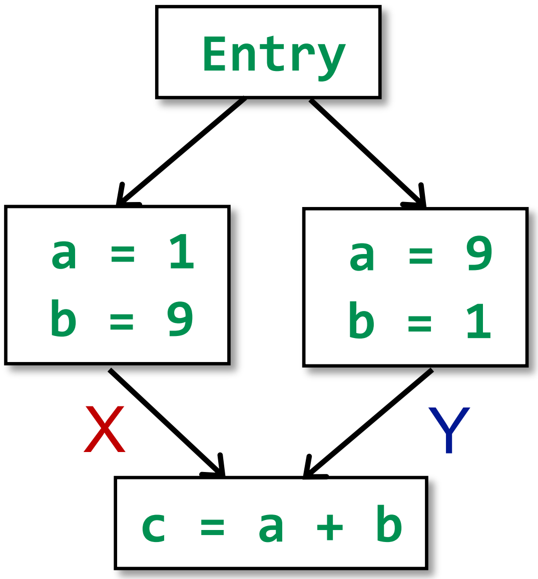

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Show our constant propagation analysis is monotonic

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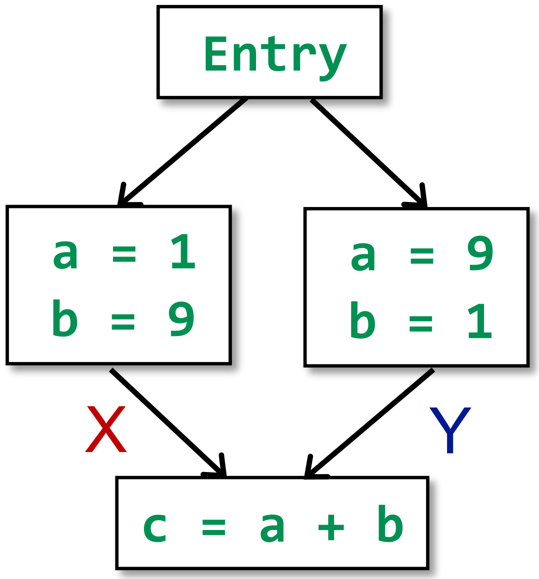

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Assignment One: Constant Propagation

Worklist Algorithm,

an optimization of Iterative Algorithm

Review Iterative Algorithm for May & Forward Analysis

INPUT: CFG ($kill_B$ and gen_B computed for each basic block B)

OUTPUT: IN[B] and OUT[B] for each basic block B

METHOD:

OUT[*entry*] = ∅;
for (each basic block *B\entry*)
    OUT[B] = ∅;
while (changes to any OUT occur)
    for (each basic block *B\entry*) {
        $\text{IN}[B] = \bigsqcup_{P \text{ a predecessor of } B} \text{OUT}[P]$;
        $\text{OUT}[B] = gen_B \cup (\text{IN}[B] - kill_B)$;
    }

Worklist Algorithm

Forward Analysis

$OUT[entry] = \emptyset;$
for (each basic block *B\entry*)
    $OUT[B] = \emptyset;$
Worklist ← all basic blocks
while (Worklist is not empty)
    Pick a basic block *B* from Worklist
    old_OUT = OUT[*B*]
    $IN[B] = \bigsqcup_{P \text{ a predecessor of } B} OUT[P];$
    $OUT[B] = gen_B \cup (IN[B] - kill_B);$
    if (old_OUT ≠ OUT[*B*])
        Add all successors of *B* to Worklist

Worklist Algorithm

Forward Analysis

$\text{OUT}[\textit{entry}] = \emptyset;$
for (each basic block $B \setminus \textit{entry}$)
    $\text{OUT}[B] = \emptyset;$
Worklist ← all basic blocks
while (Worklist is not empty)
    Pick a basic block B from Worklist
    old_OUT = OUT[B]
    $\text{IN}[B] = \bigsqcup_{P \textit{ a predecessor of } B} \text{OUT}[P];$
    $\text{OUT}[B] = \textit{gen}_B \cup (\text{IN}[B] - \textit{kill}_B);$
    if (old_OUT $\neq$ OUT[B])
        Add all successors of B to Worklist

OUT will not change if IN does not change

Summary (I)

1. Iterative Algorithm, Another View
2. Partial Order
3. Upper and Lower Bounds
4. Lattice, Semilattice, Complete and Product Lattice
5. Data Flow Analysis Framework via Lattice
6. Monotonicity and Fixed Point Theorem

7. Relate Iterative Algorithm to Fixed Point Theorem

8. May/Must Analysis, A Lattice View

9. MOP and Distributivity

10. Constant Propagation

11. Worklist Algorithm

Summary (II)

The X You Need To Understand in This Lecture

- Understand the functional view of iterative algorithm
- The definitions of lattice and complete lattice
- Understand the fixed-point theorem
- How to summarize may and must analyses in lattices
- The relation between MOP and the solution produced by the iterative algorithm
- Constant propagation analysis
- Worklist algorithm

软件分析

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程序设计语言与
静态分析研究组
李樾 谭添