软件分析

南京大学

计算机科学与技术系

程序设计语言与静态分析研究组

李樾 谭添

Static Program Analysis

Data Flow Analysis — Foundations

Nanjing University

Yue Li

2021

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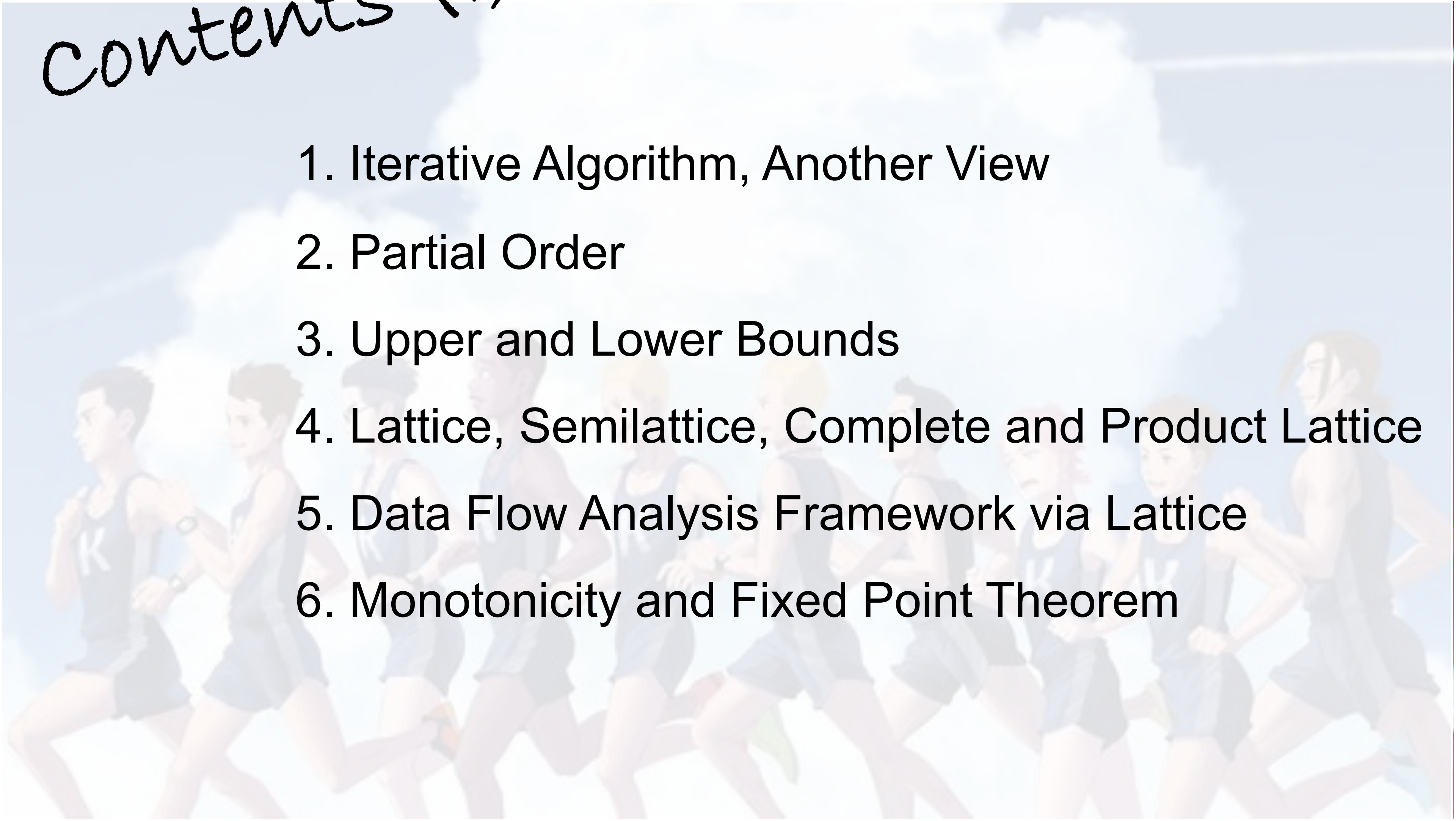

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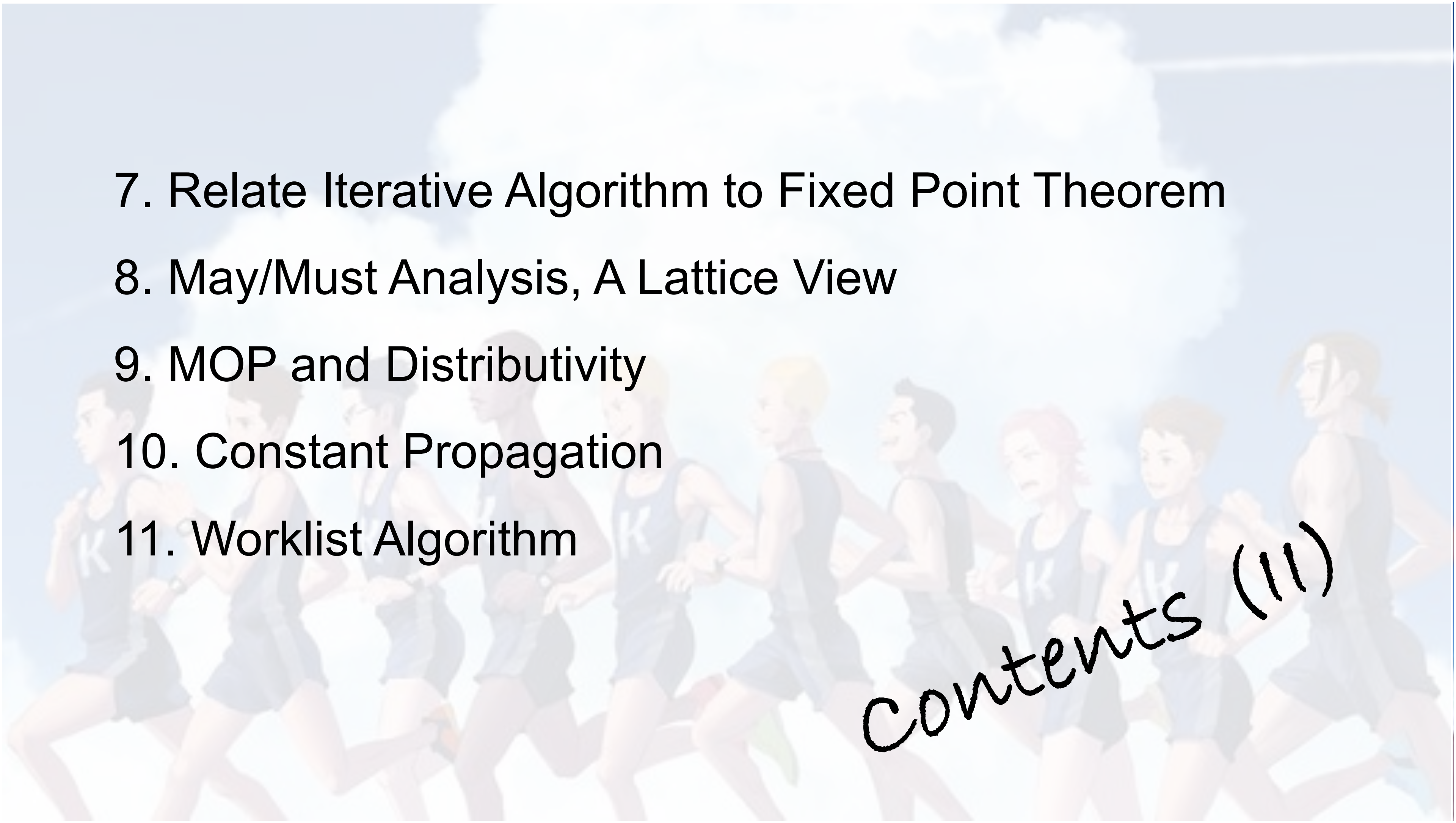

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Let us first recall the iterative algorithm for data flow analysis

This general iterative algorithm produces a solution to data flow analysis

Iterative Algorithm for May & Forward Analysis

INPUT: CFG ($kill_B$ and gen_B computed for each basic block *B*)

OUTPUT: IN[*B*] and OUT[*B*] for each basic block *B*

METHOD:

```
OUT[entry] = $\emptyset$;
for (each basic block B\entry)
    OUT[B] = $\emptyset$;
while (changes to any OUT occur)
    for (each basic block B\entry) {
        IN[B] = $\bigcup_{P \text{ a predecessor of } B}$ OUT[P];
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$$(\mathrm{OUT}[n_1], \mathrm{OUT}[n_2], \ldots, \mathrm{OUT}[n_k])$$

as an element of set $(V_1 \times V_2 \ldots \times V_k)$ denoted as V^k , to hold the values of the analysis after each iteration.

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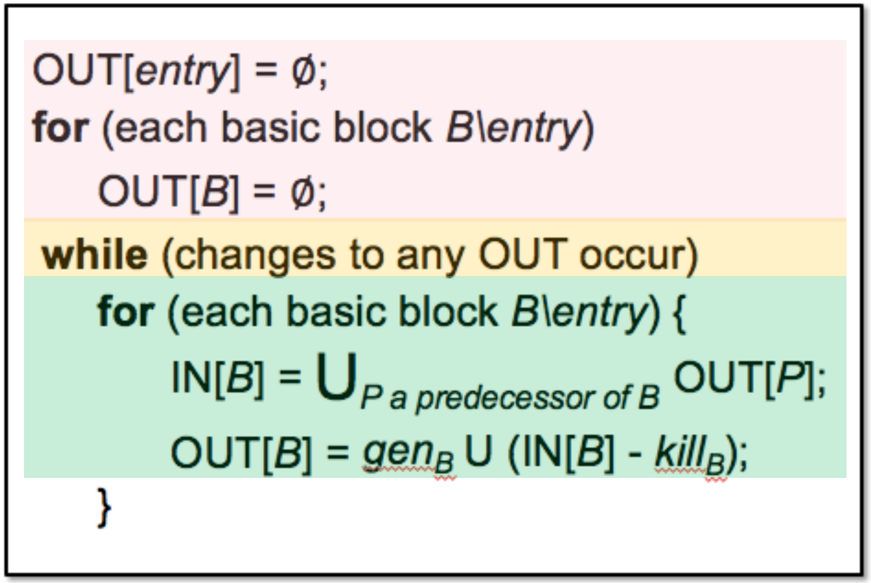
- Each iteration can be considered as taking an action to map an element of V^k to a new element of V^k, through applying the transfer functions and control-flow handing, abstracted as a function $F: V^k \to V^k$
- Then the algorithm outputs a series of k-tuples iteratively until a k-tuple is the same as the last one in two consecutive iterations



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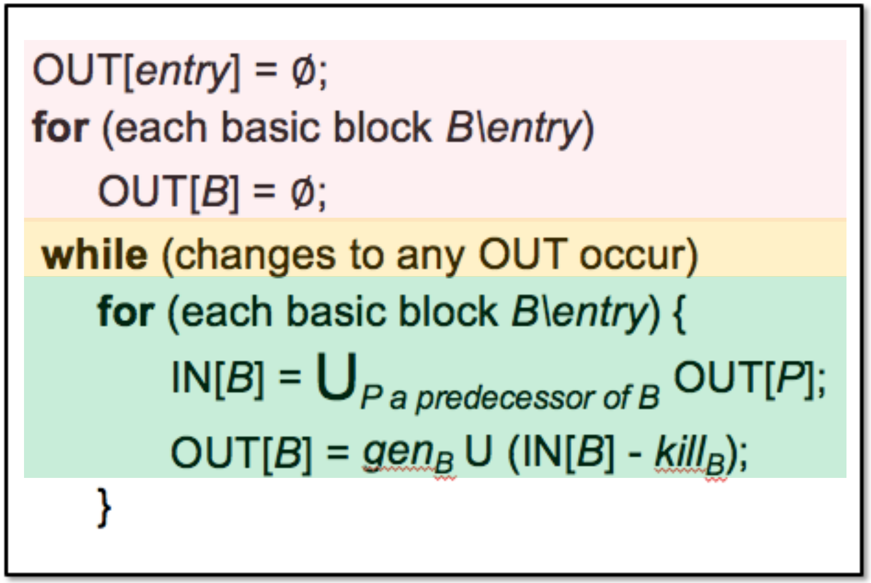

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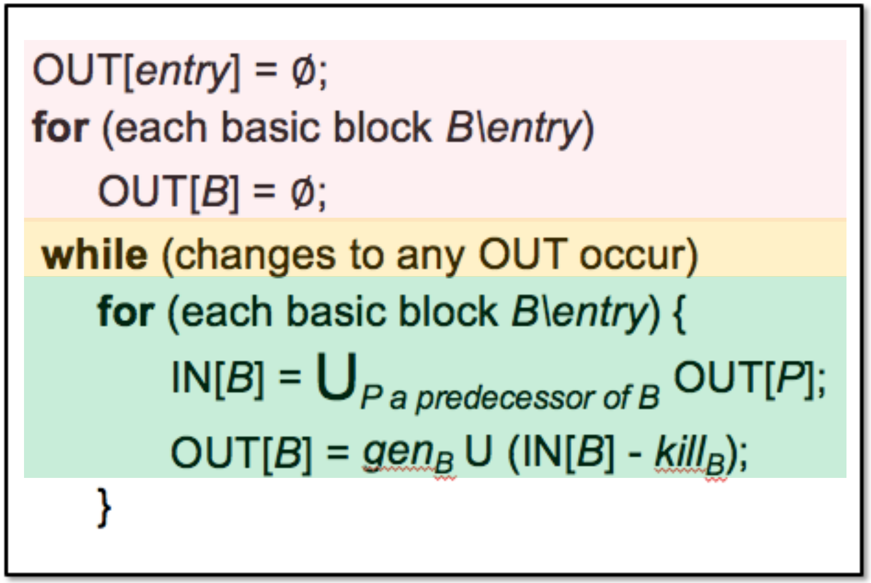

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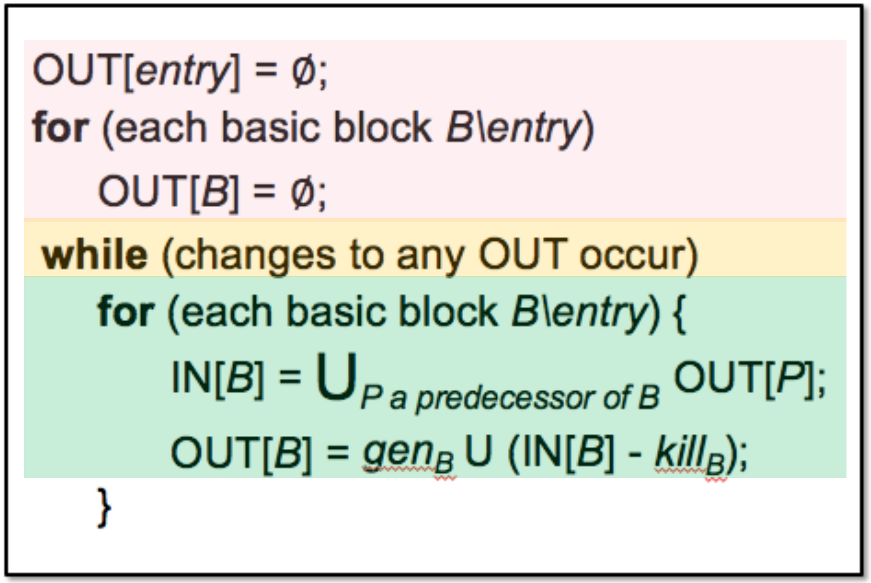

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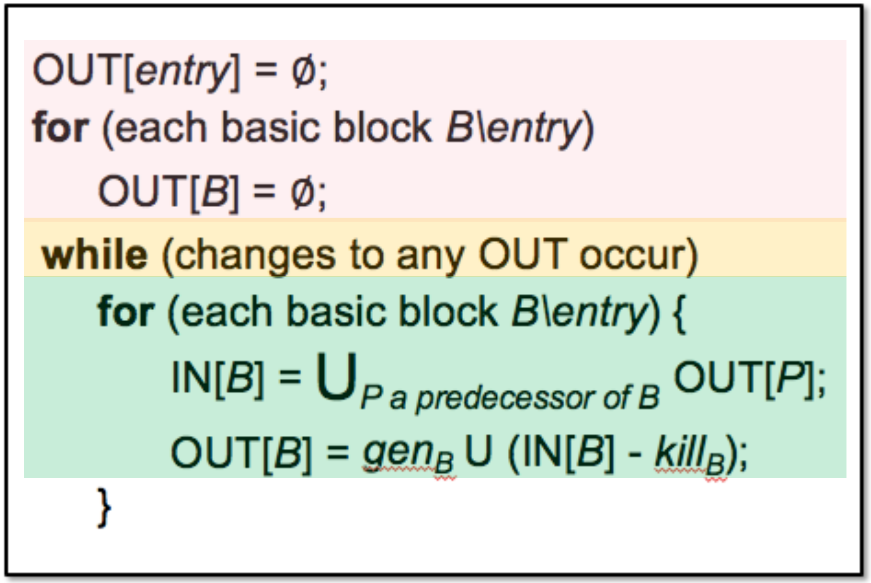

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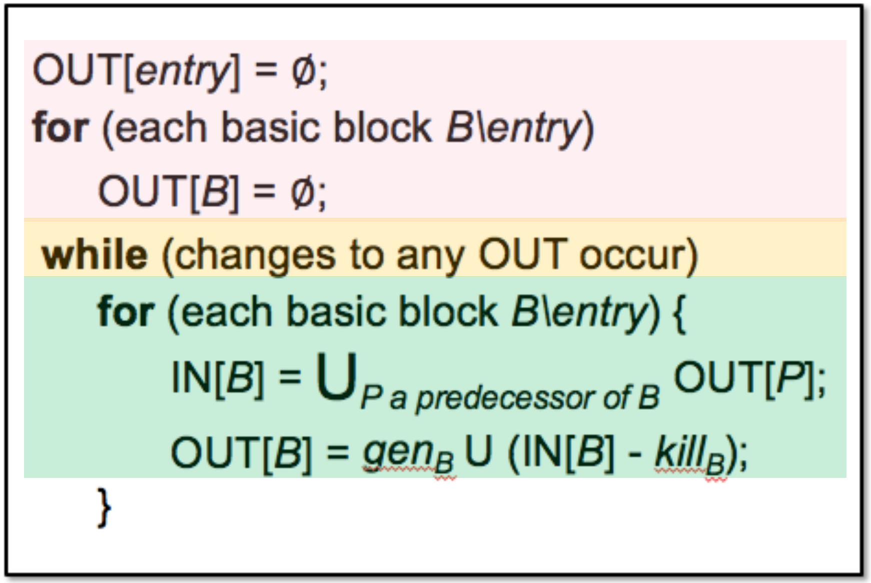

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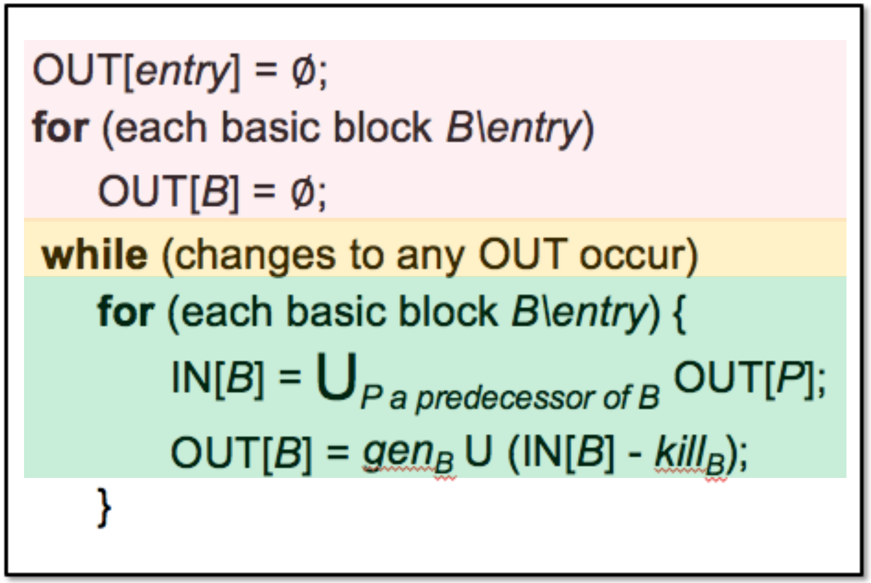

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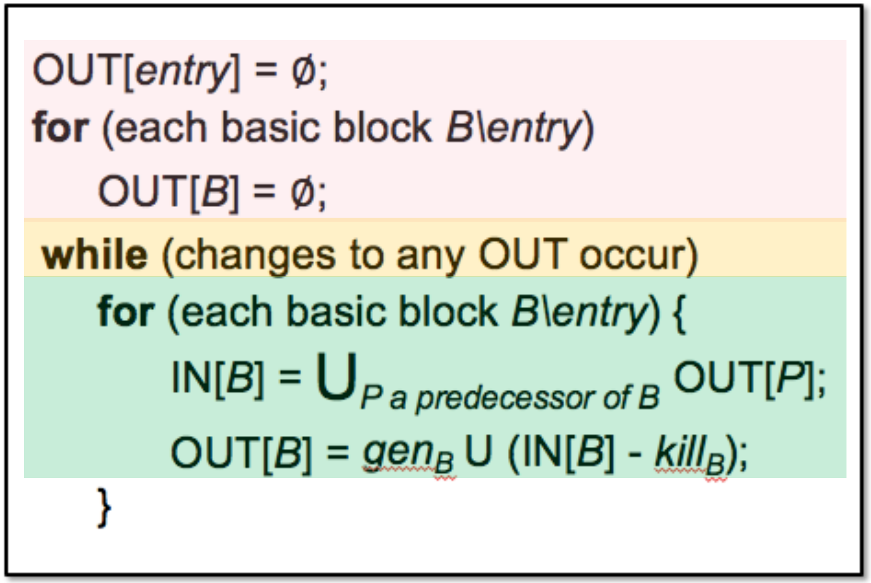

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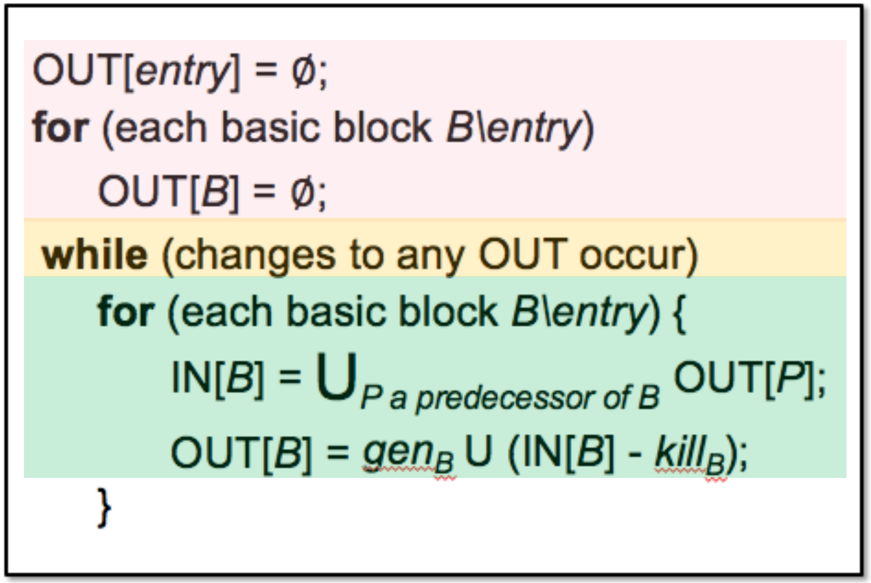

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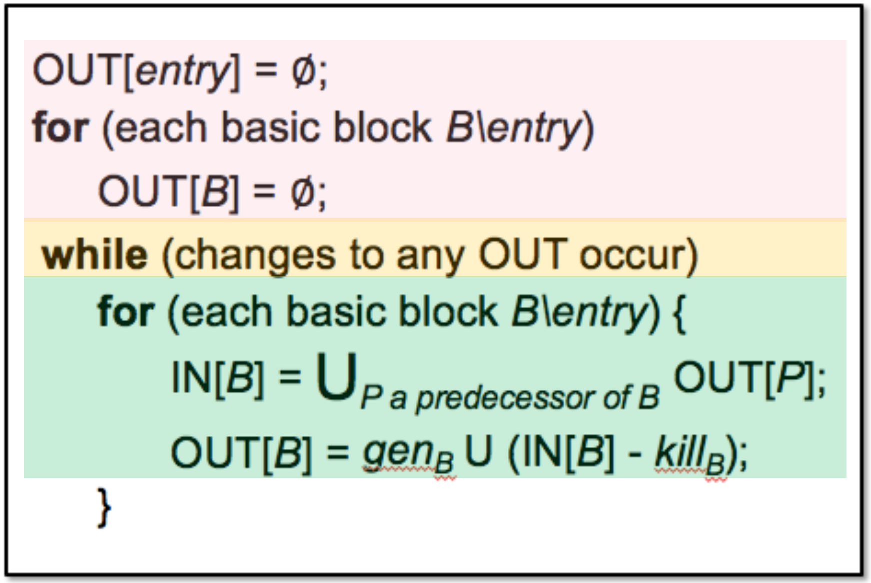

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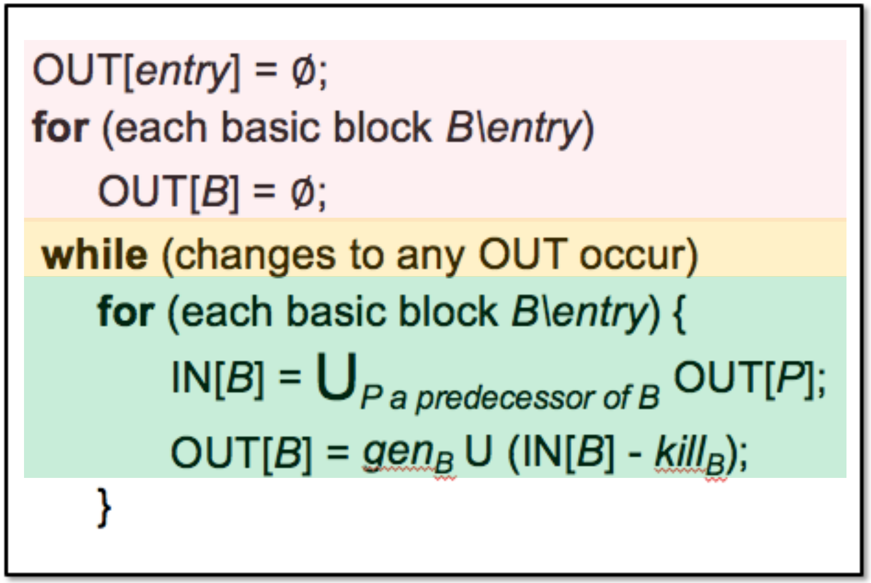

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⋮

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The iterative algorithm reaches a fixed point

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(3) *Transitivity* $1 \leq 2 \land 2 \leq 3$ then $1 \leq 3$

Partial Order

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Example 1. Is $(S, \sqsubseteq)$ a poset where S is a set of integers and $\sqsubseteq$ represents $\leq$ (less than or equal to)?

✓ (1) *Reflexivity* $1 \leq 1, 2 \leq 2$

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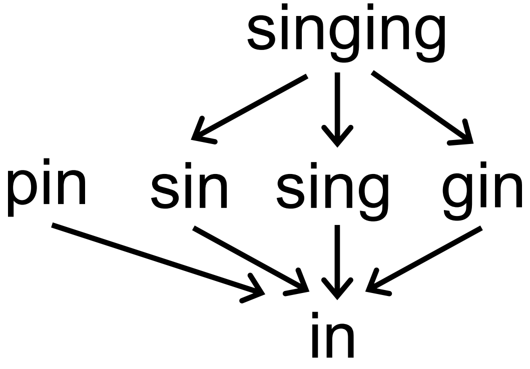

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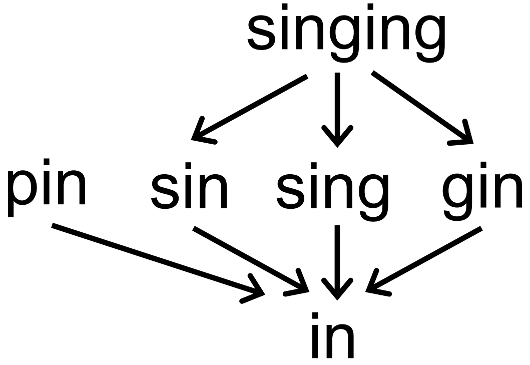

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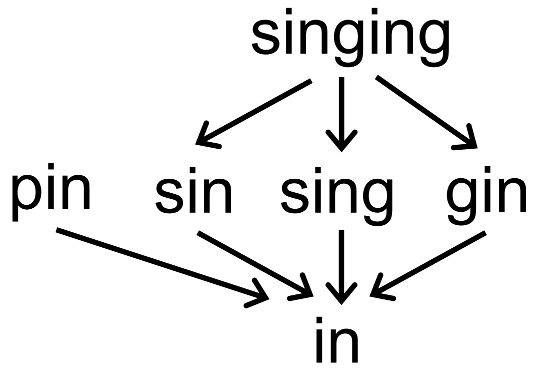

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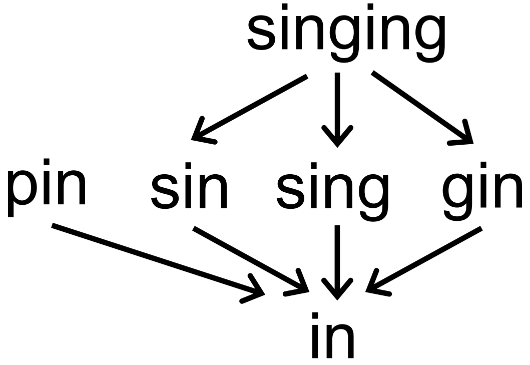

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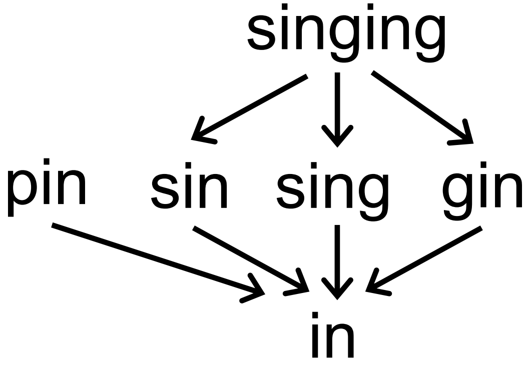

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partial means for a pair of set elements in P, they could be incomparable; in other words, not necessary that every pair of set elements must satisfy the ordering $\sqsubseteq$

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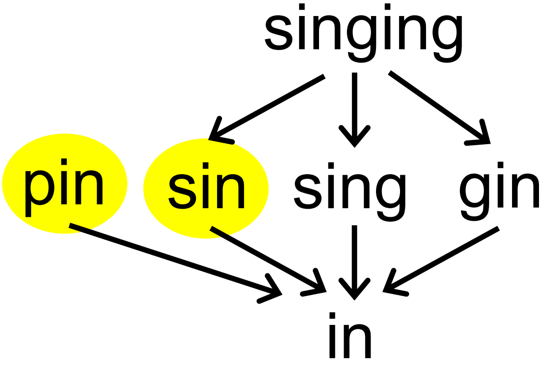

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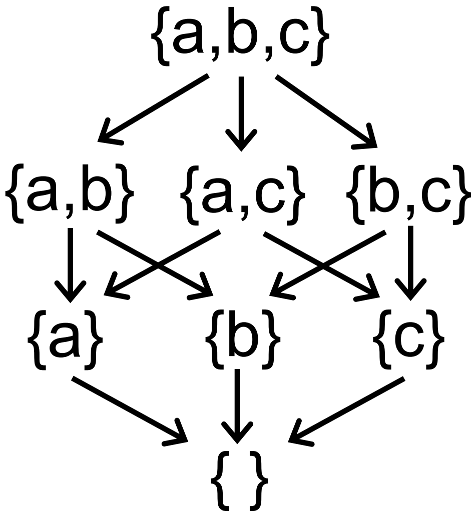

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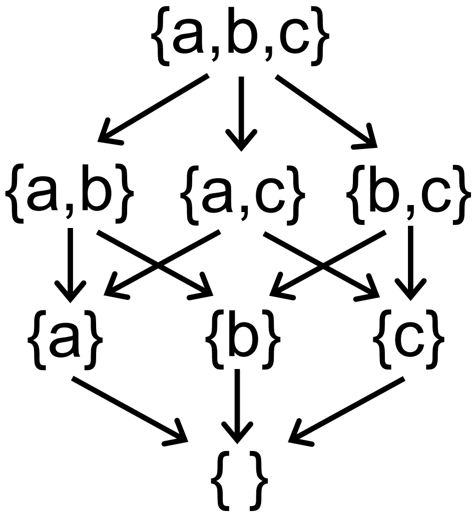

{a,b,c}

{a,b} {a,c} {b,c}

{a} {b} {c}

{ }

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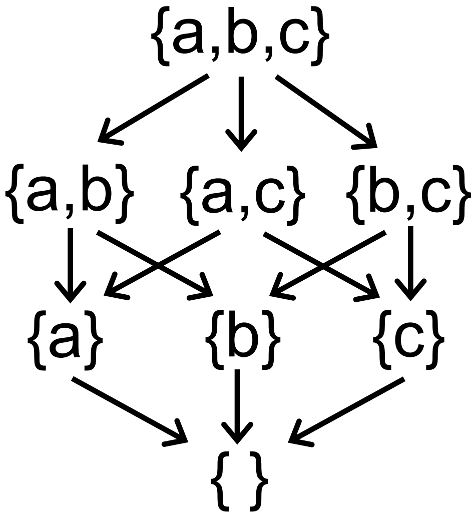

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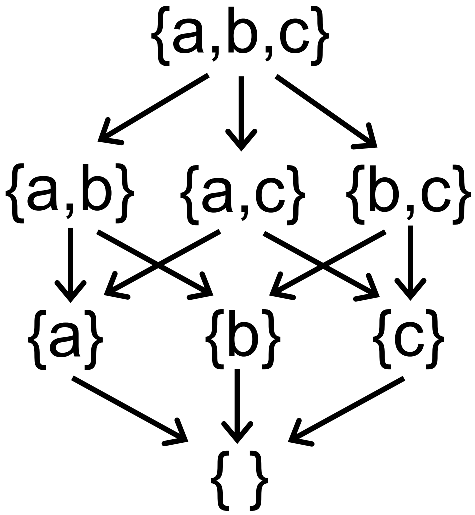

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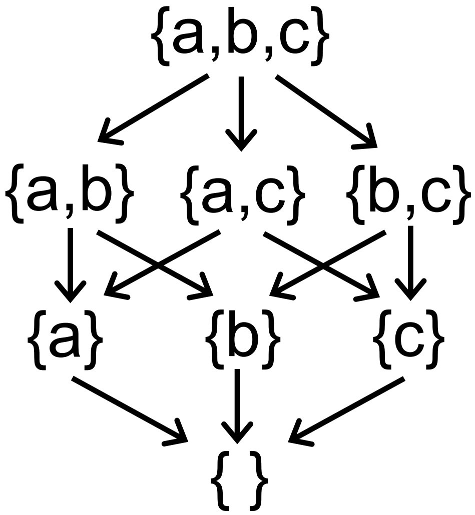

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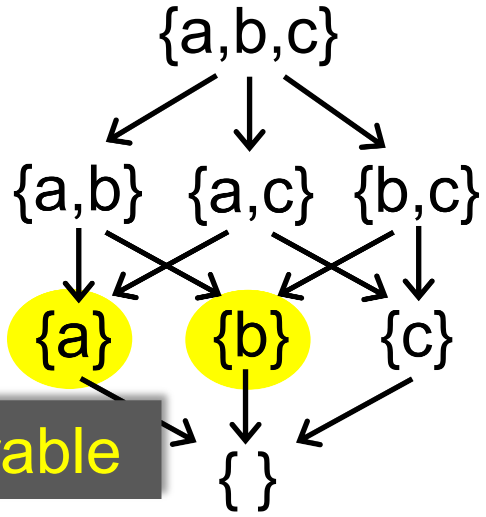

partial → incomparable

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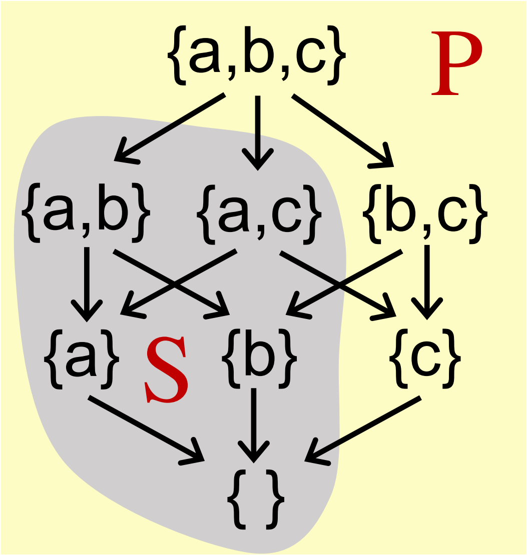

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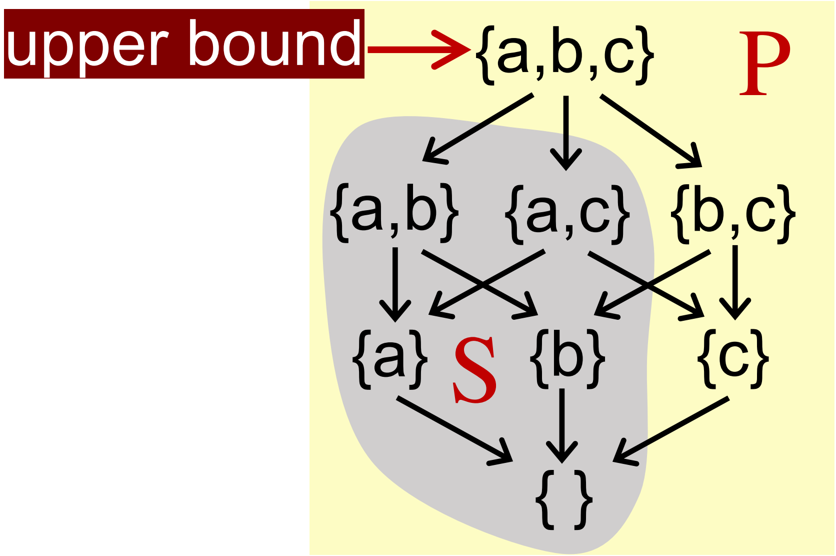

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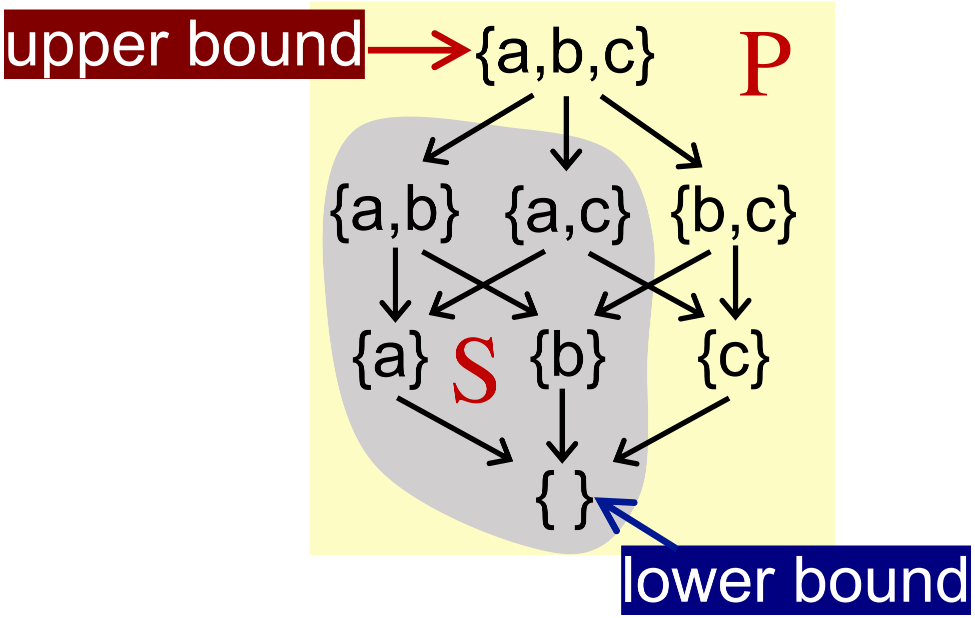

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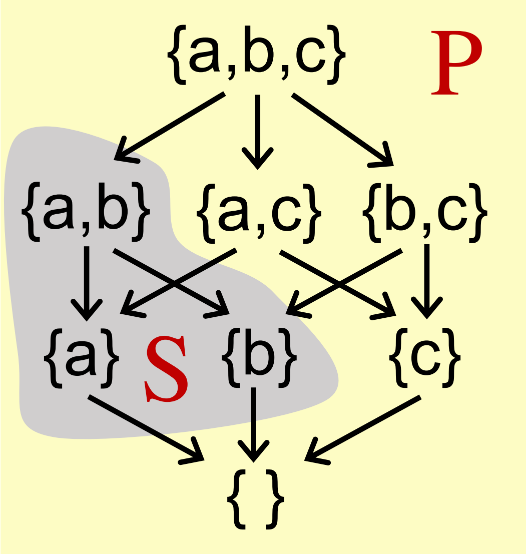

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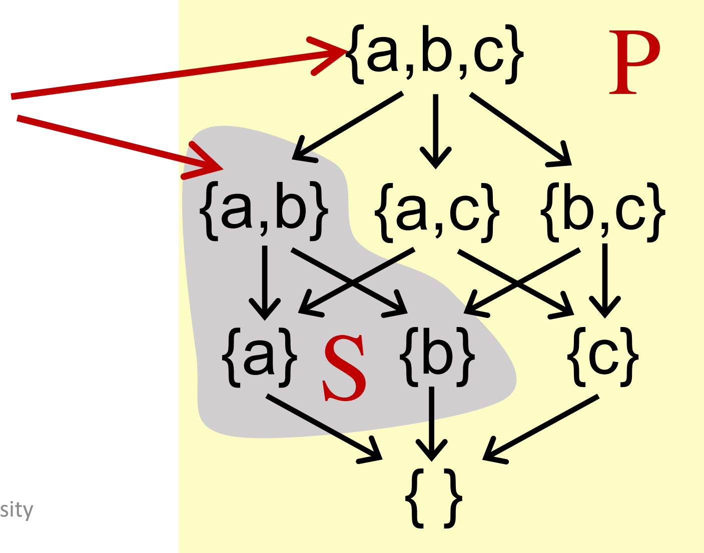
upper bound



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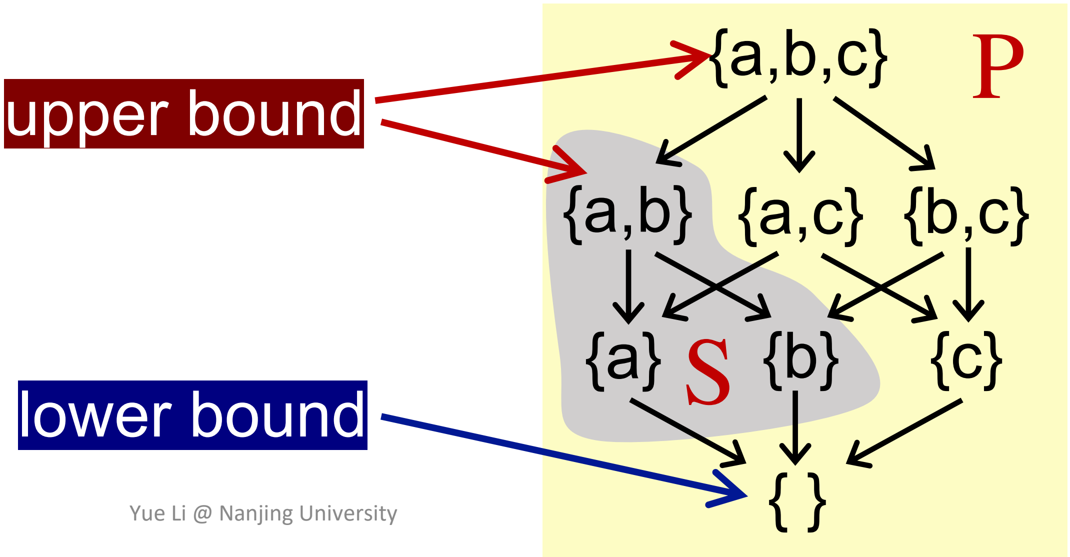

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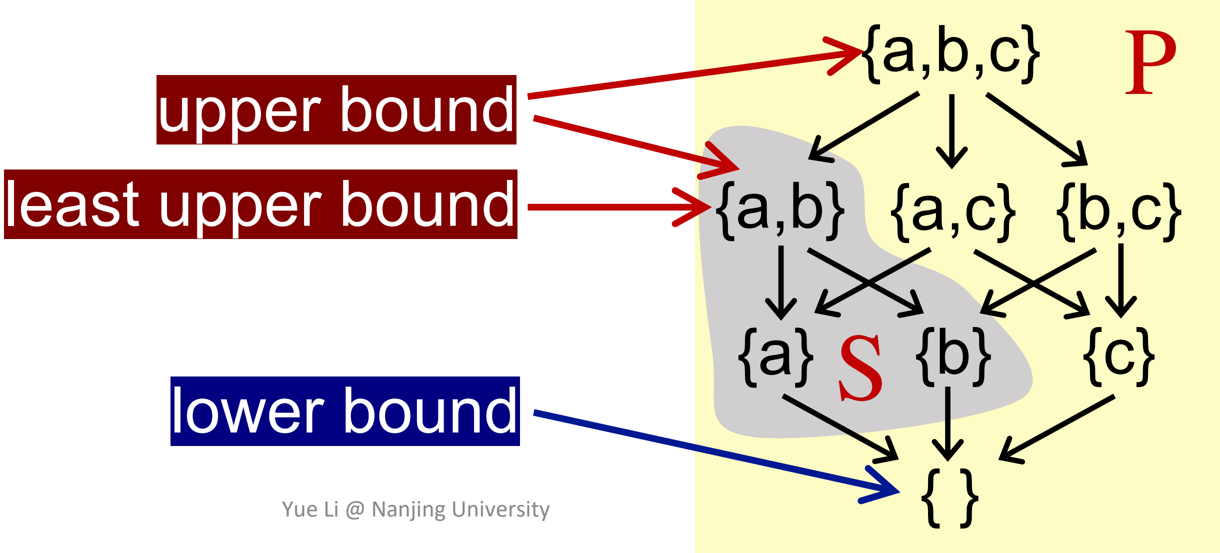

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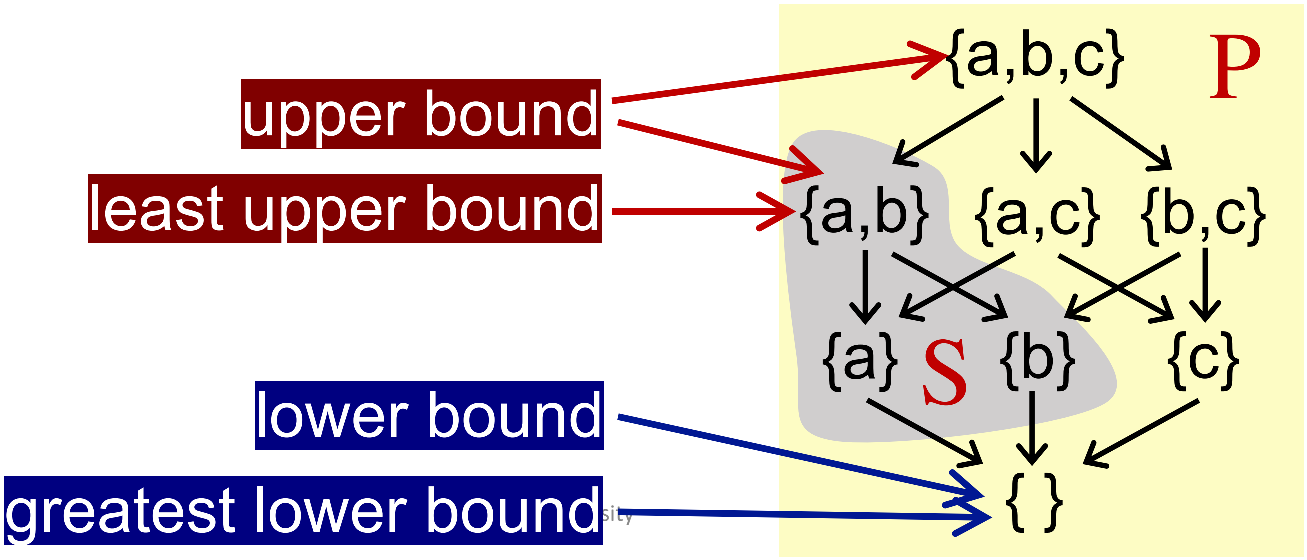

upper bound

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Usually, if S contains only two elements a and b ($S = \{a, b\}$), then $\sqcup S$ can be written $a \sqcup b$ (the join of a and b)
$\sqcap S$ can be written $a \sqcap b$ (the meet of a and b)

Some Properties

- Not every poset has *lub* or *glb*

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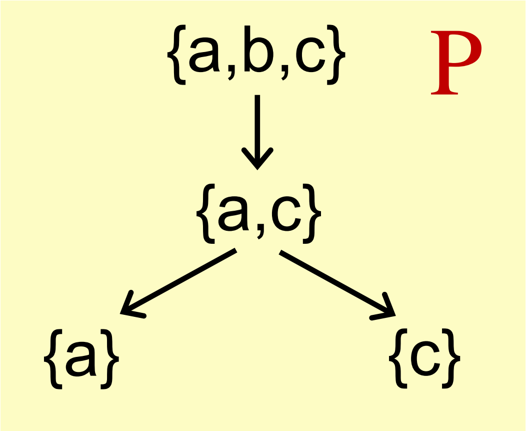

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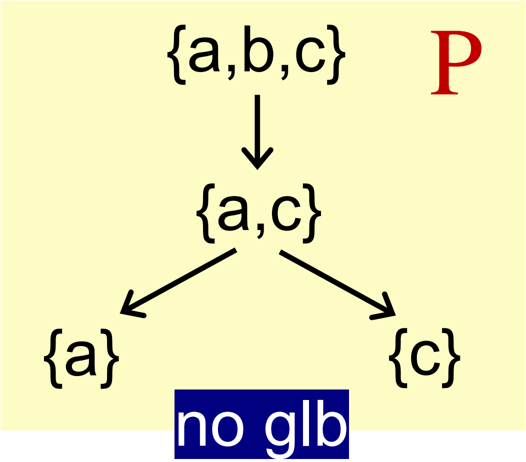

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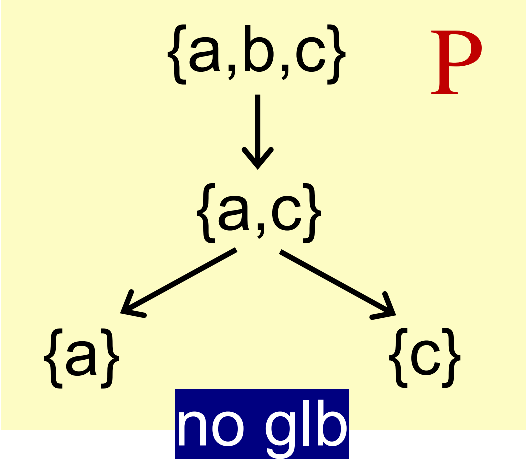

P

{a,b,c}

{a,c}

{a}

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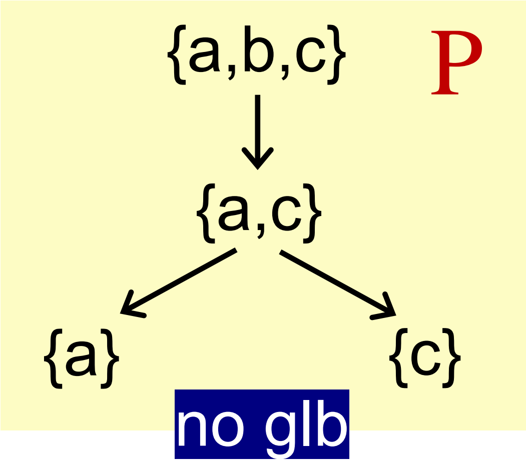

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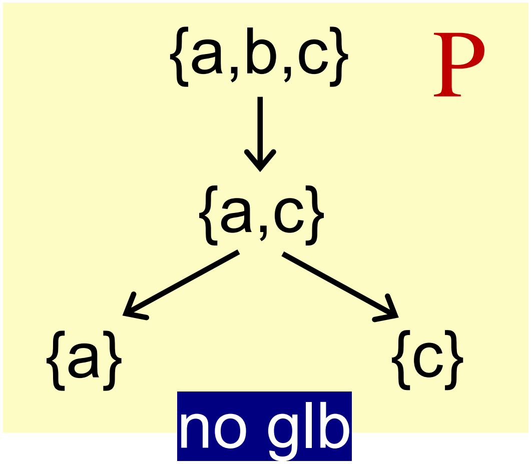

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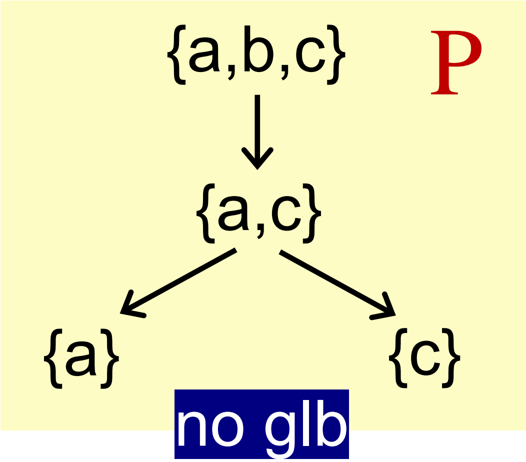

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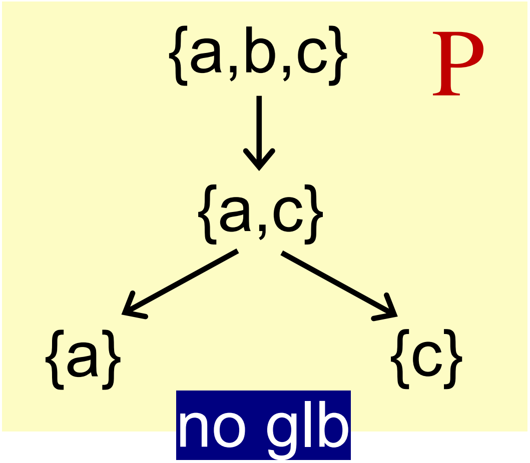

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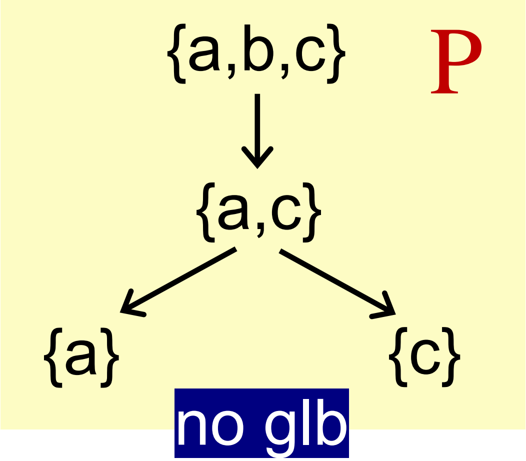

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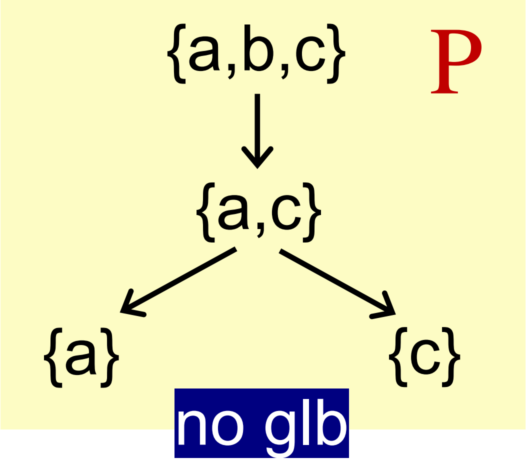

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$g_1 = g_2$

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✓ The ⊔ operator means "max" and ⊓ operator means "min"

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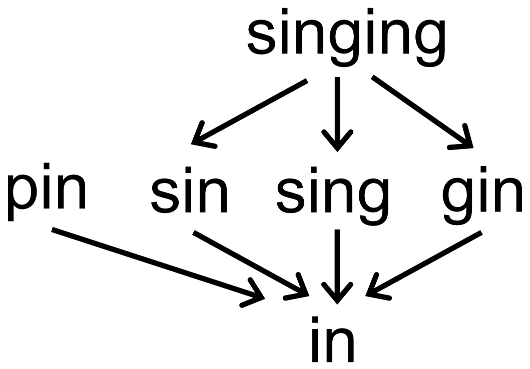

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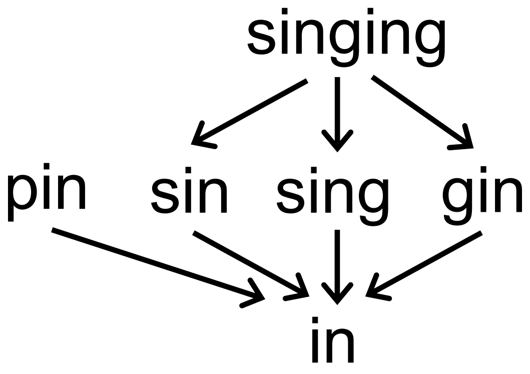

singing

pin sin sing gin

in

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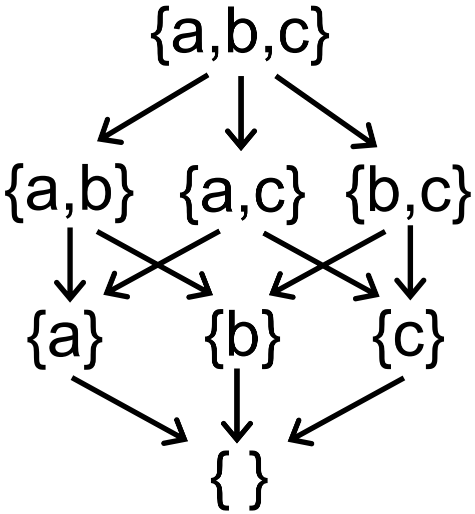

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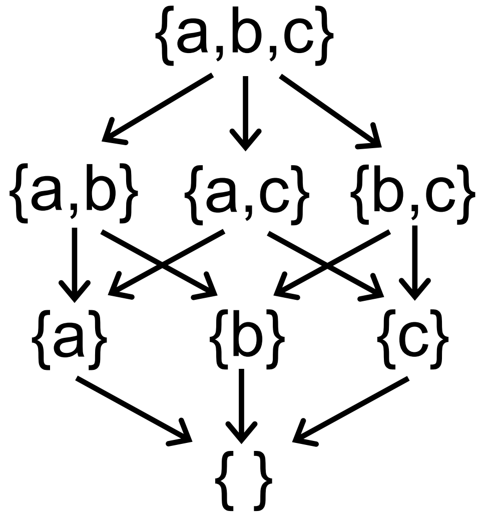

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Semilattice

Given a poset $(P, \sqsubseteq)$, $\forall a, b \in P$,
if only $a \sqcup b$ exists, then $(P, \sqsubseteq)$ is called a join semilattice
if only $a \sqcap b$ exists, then $(P, \sqsubseteq)$ is called a meet semilattice

Complete Lattice

Given a lattice (P, $\sqsubseteq$), for arbitrary subset S of P, if $\sqcup$S and $\sqcap$S exist, then (P, $\sqsubseteq$) is called a complete lattice

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✗ For a subset S^+ including all positive integers, it has no $\sqcup S^+$ $(+\infty)$

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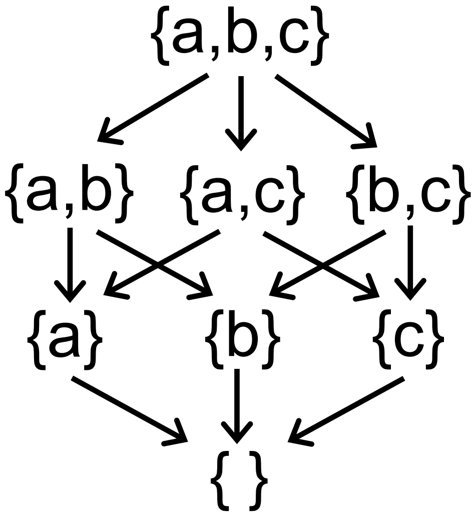

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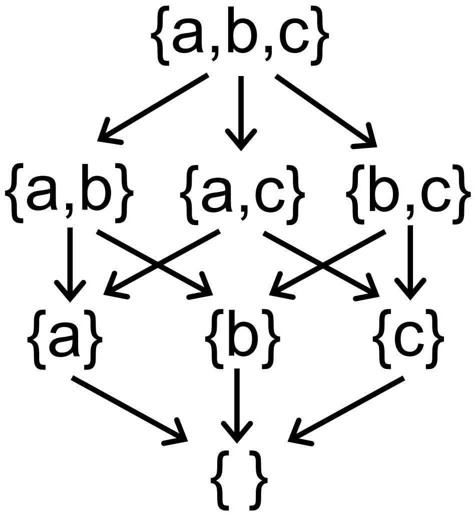
✓ Note: the definition of bounds implies that the bounds are not necessarily in the subsets (but they must be in the lattice)



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a greatest element $\top = \sqcup P$ called top and
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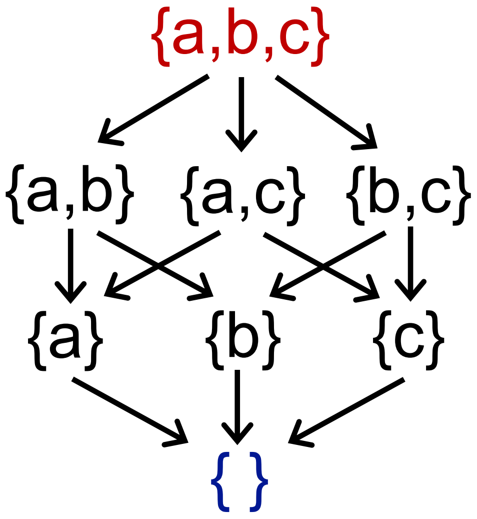

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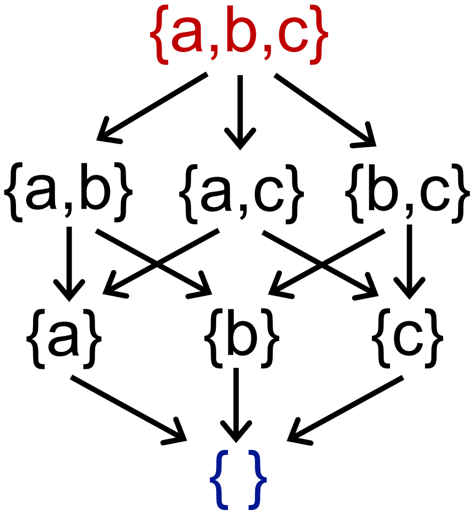

Complete Lattice Mostly focused in data flow analysis

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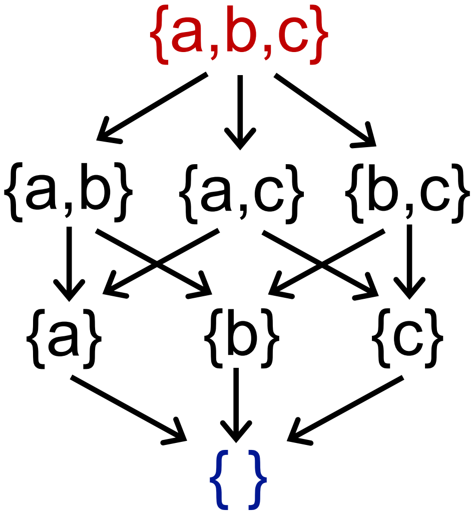

{a,b,c}

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{a} {b} {c}

{ }

Product Lattice

Given lattices $L_1 = (P_1, \sqsubseteq_1)$, $L_2 = (P_2, \sqsubseteq_2)$, …, $L_n = (P_n, \sqsubseteq_n)$, if for all i, $(P_i, \sqsubseteq_i)$ has $\sqcup_i$ (least upper bound) and $\sqcap_i$ (greatest lower bound), then we can have a product lattice $L^n = (P, \sqsubseteq)$ that is defined by:

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Data Flow Analysis Framework via Lattice

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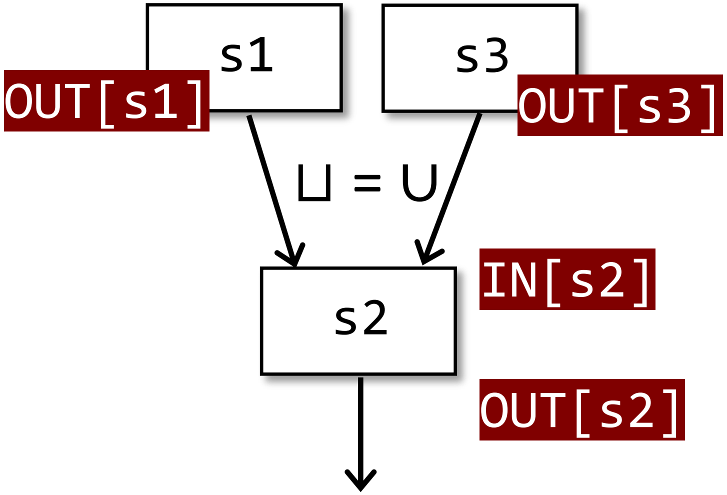

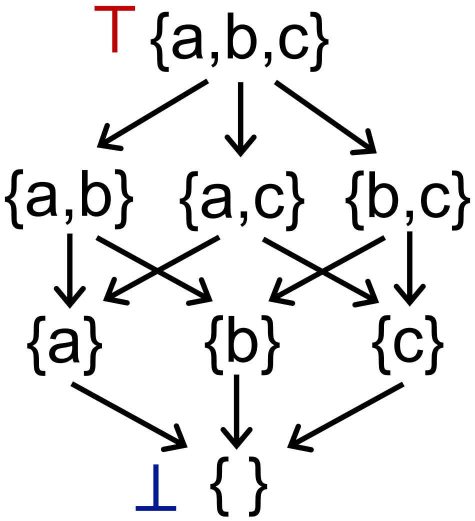

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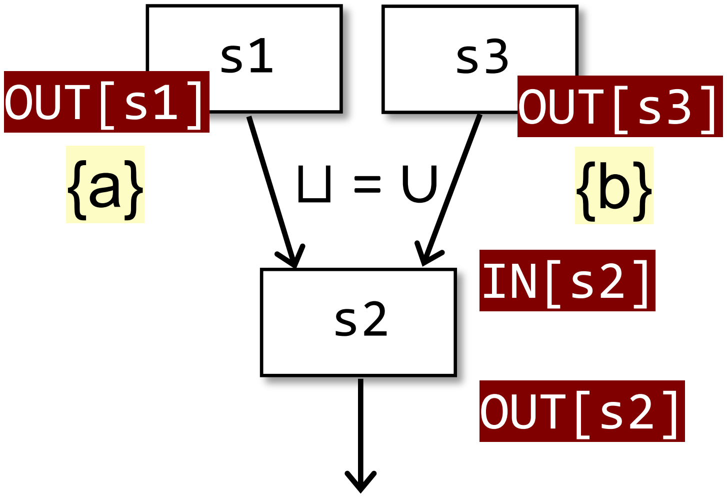

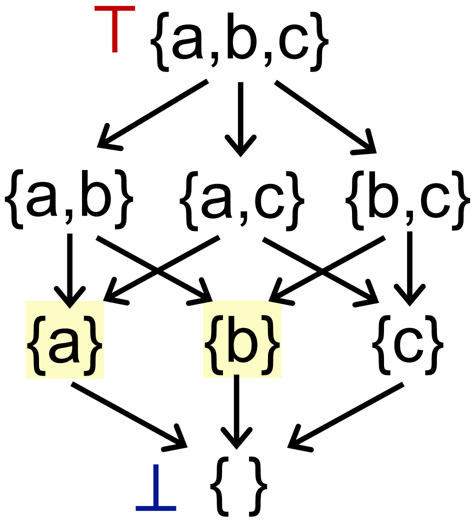

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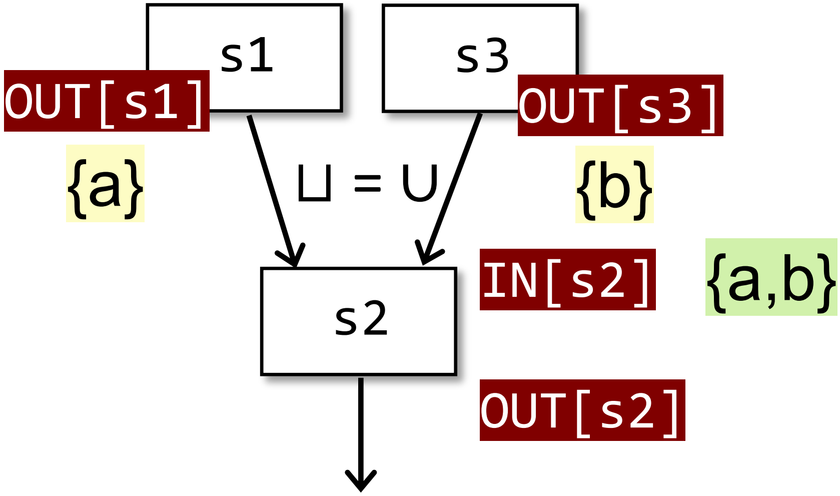

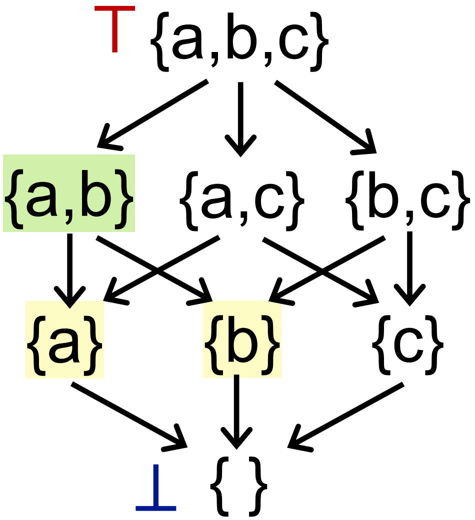

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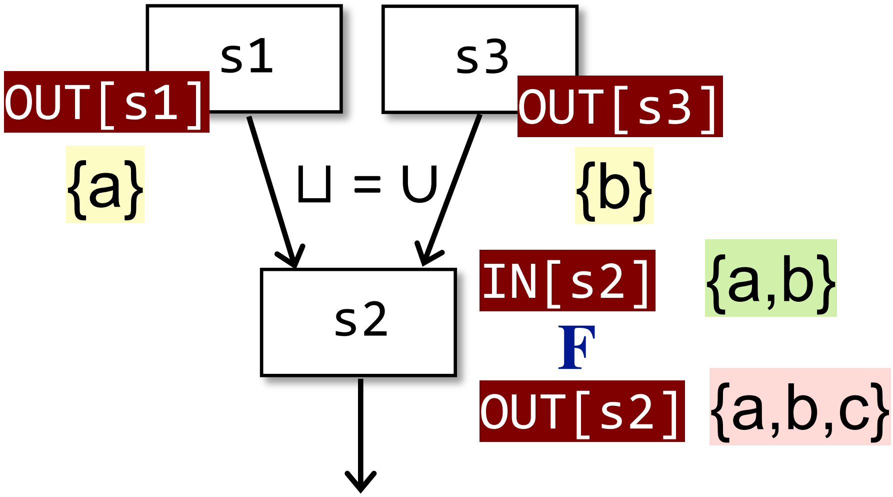

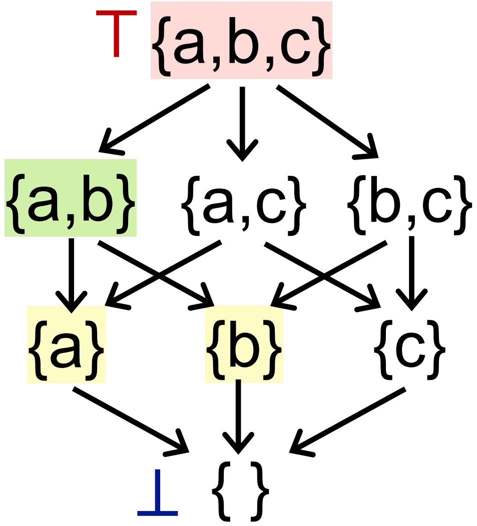

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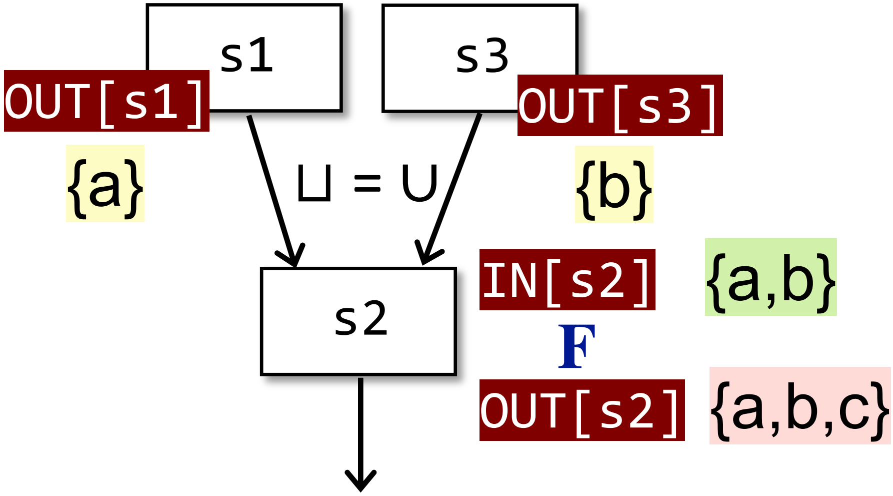

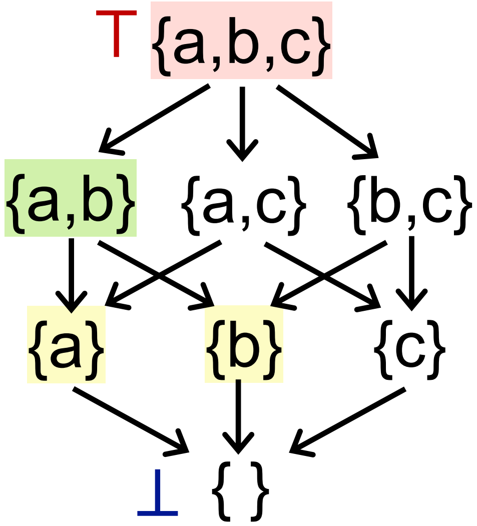

Data flow analysis can be seen as iteratively applying transfer functions and meet/join operations on the values of a lattice

Review The Questions We Have Seen Before

The iterative algorithm (or the IN/OUT equation system) produces a solution to a data flow analysis

- Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?

- If so, is there only one solution or only one fixed point? If more than one, is our solution the best one (most precise)?

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- ? If so, is there only one solution or only one fixed point? If more than one, is our solution the best one (most precise)?
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Review The Questions We Have Seen Before

The iterative algorithm (or the IN/OUT
produces a solution to a da

Recall "OUT never shrinks"
It is about monotonicity

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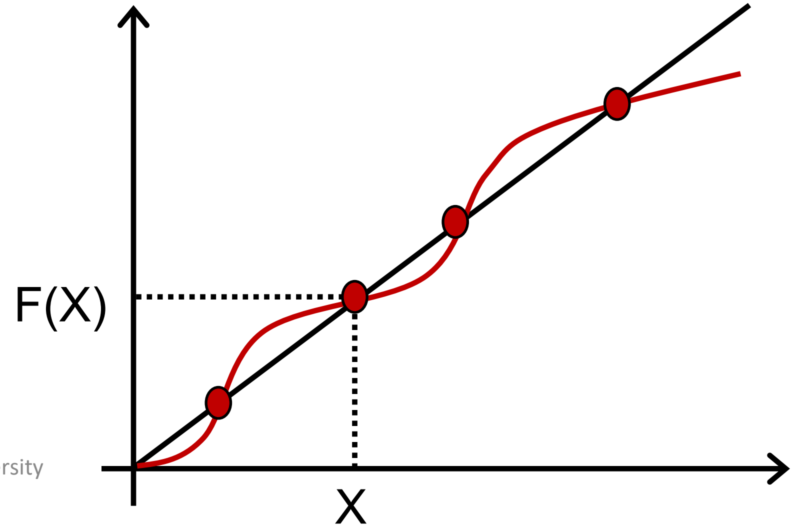

Monotonicity

A function $f: L \to L$ (L is a lattice) is monotonic if $\forall x, y \in L$,

$$x \sqsubseteq y \Longrightarrow f(x) \sqsubseteq f(y)$$

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$f(\top$ …, $f^k(\top)$ until a fixed point is reached

Let us prove

(1) Existence of fixed point

(2) The fixed point is the least

Fixed-Point Theorem (Existence of Fixed Point)

Proof:

By the definition of $\perp$ and f: L → L, we have

$$\perp \sqsubseteq f(\perp)$$

Fixed-Point Theorem (Existence of Fixed Point)

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As f is monotonic, we have

$$f(\perp) \sqsubseteq f(f(\perp)) = f^2(\perp)$$

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$$\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \ \ldots \sqsubseteq f^i(\perp)$$

As L is finite (its height is H), the values are bounded among

$$\perp , f(\perp) , f^2(\perp) \ldots f^H(\perp)$$

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As L is finite (its height is H), the values are bounded among

$$\bot\ ,\ f(\bot)\ ,\ f^2(\bot)\ \dots\ f^H(\bot)$$

When $i > H$, by pigeonhole principle, there exists k and j that

$$f^k(\bot) = f^j(\bot) \text{ (assume } k < j \leq H+1)$$

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Further as $f^k(\bot) \sqsubseteq \dots \sqsubseteq f^j(\bot)$, we have

$$f^{Fix} = f^k(\bot) = f^{k+1}(\bot) = f^j(\bot)$$

Fixed-Point Theorem (Existence of Fixed Point)

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$$\perp \sqsubseteq f(\perp)$$

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Further as $f^k(\perp) \sqsubseteq \ldots \sqsubseteq f^j(\perp)$, we have

$$f^{Fix} = f^k(\perp) = f^{k+1}(\perp) = f^j(\perp)$$

Thus, the fixed point exists.

Fixed-Point Theorem (Least Fixed Point)

Proof:

Assume we have another fixed point x, i.e., x = f(x)

Fixed-Point Theorem (Least Fixed Point)

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By the definition of $\bot$, we have $\bot \sqsubseteq x$

Induction begins:

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Induction begins:

As f is monotonic, we have

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Proof:

Assume we have another fixed point x, i.e., $x = f(x)$

By the definition of $\bot$, we have $\bot \sqsubseteq x$

Induction begins:

As f is monotonic, we have

$$f(\bot) \sqsubseteq f(x)$$

Assume $f^i(\bot) \sqsubseteq f^i(x)$, as f is monotonic, we have

$$f^{i+1}(\bot) \sqsubseteq f^{i+1}(x)$$

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Thus $f^i(\bot) \sqsubseteq f^i(x) = x$, then we have

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Thus the fixed point is the least

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The proof for greatest fixed point is similar

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Review The Questions We Have Seen Before

The iterative algorithm (or the IN/OUT equation system) produces a solution to a data flow analysis

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greatest or least fixed point

- When will the algorithm reach the fixed p can we get the solution?

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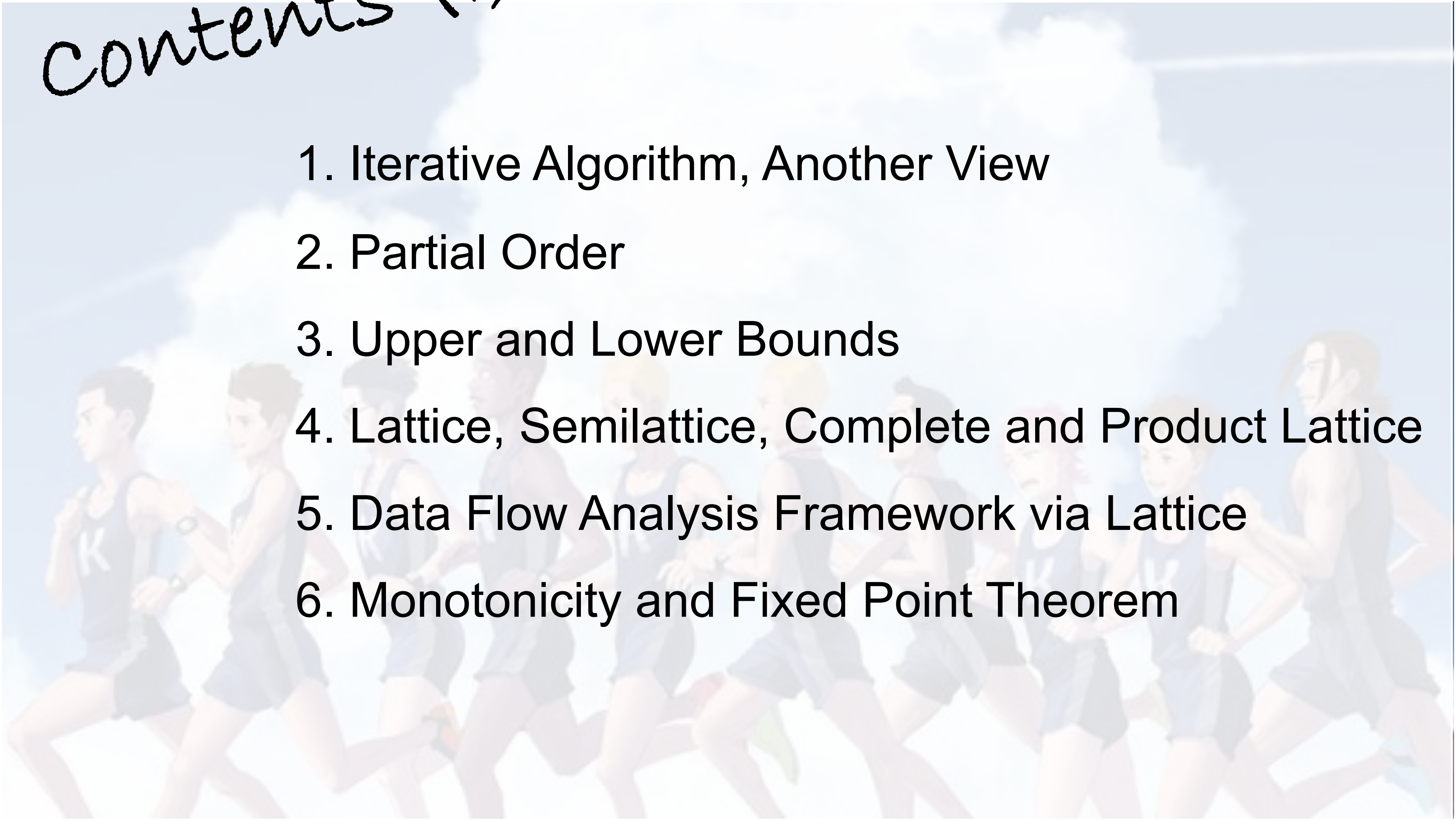
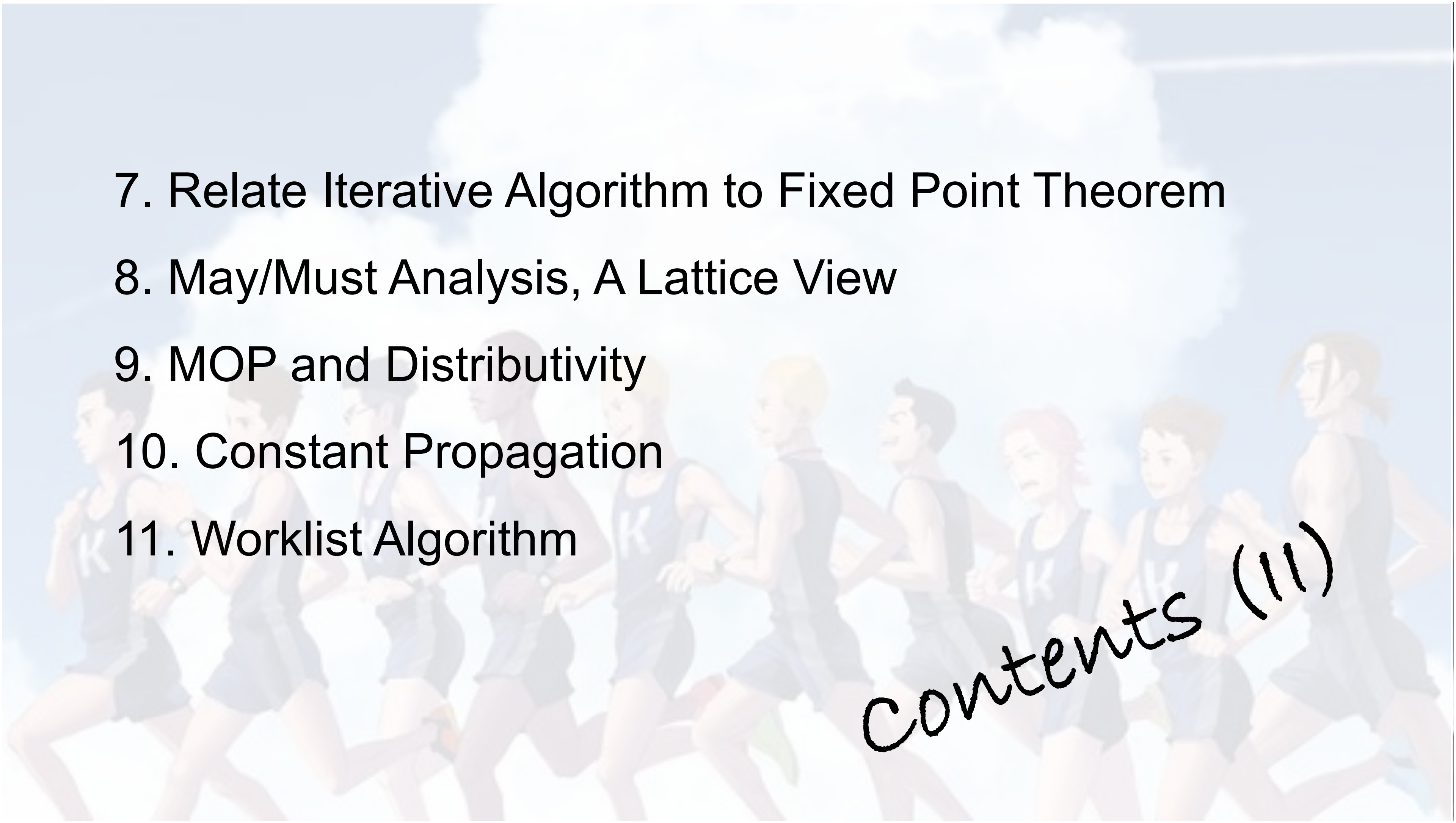
Now what we have just seen is the property (fixed point theorem) for the function on a lattice. We cannot say our iterative algorithm also has that property unless we can *relate the algorithm to the fixed point theorem*, if possible

Contents (1)



1. Iterative Algorithm, Another View
2. Partial Order
3. Upper and Lower Bounds
4. Lattice, Semilattice, Complete and Product Lattice
5. Data Flow Analysis Framework via Lattice
6. Monotonicity and Fixed Point Theorem



7. Relate Iterative Algorithm to Fixed Point Theorem

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Contents (II)

Static Program Analysis

Data Flow Analysis — Foundations

Nanjing University

Yue Li

2021

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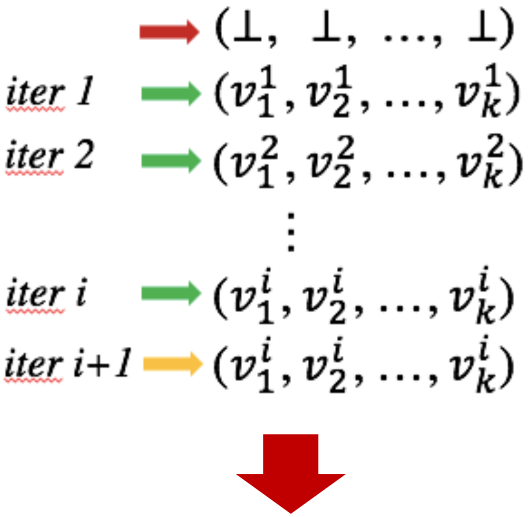
Relate Iterative Algorithm to Fixed-Point Theorem



$\rightarrow (\bot,\ \bot,\ \ldots,\ \bot)$

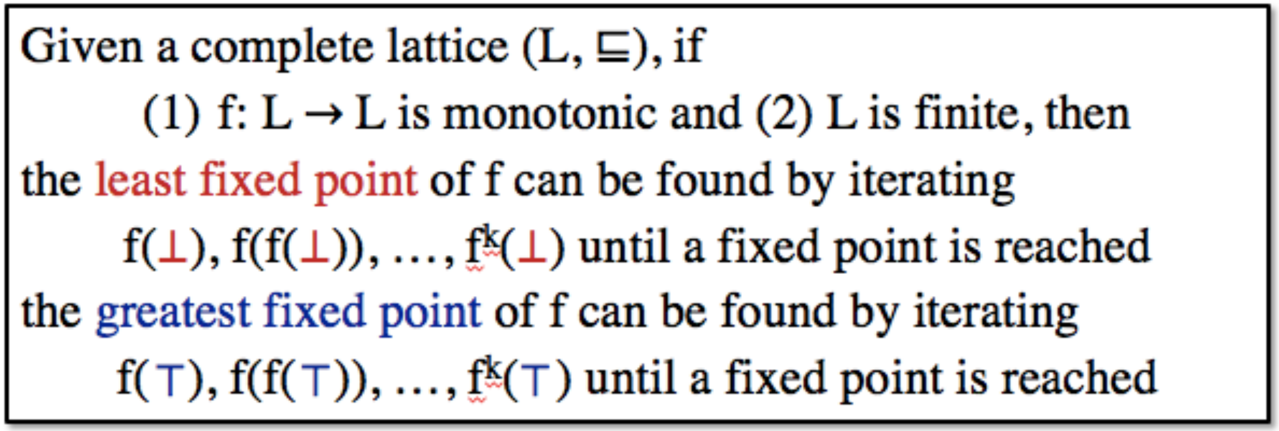
iter 1 $\rightarrow (v_1^1, v_2^1, \ldots, v_k^1)$

iter 2 $\rightarrow (v_1^2, v_2^2, \ldots, v_k^2)$

$\vdots$

iter i $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$

iter i+1 $\rightarrow (v_1^i, v_2^i, \ldots, v_k^i)$



Given a complete lattice (L, ⊑), if

(1) f: L → L is monotonic and (2) L is finite, then

the least fixed point of f can be found by iterating

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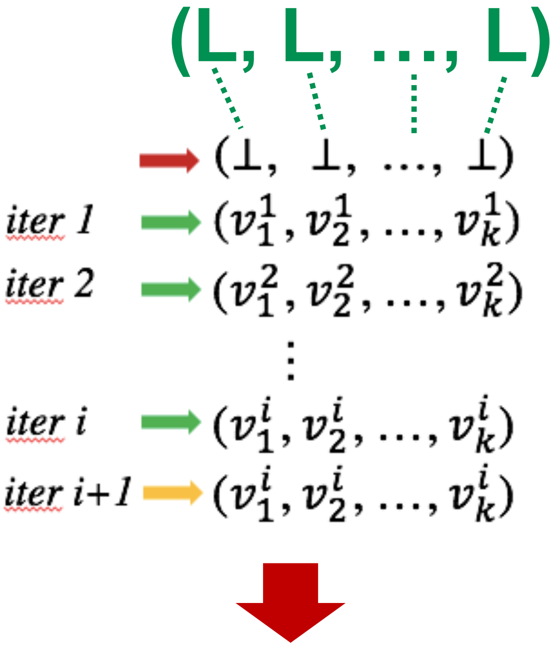
Relate Iterative Algorithm to Fixed-Point Theorem



(L, L, …, L)

$(\bot, \bot, \ldots, \bot)$

iter 1 $(v_1^1, v_2^1, \ldots, v_k^1)$

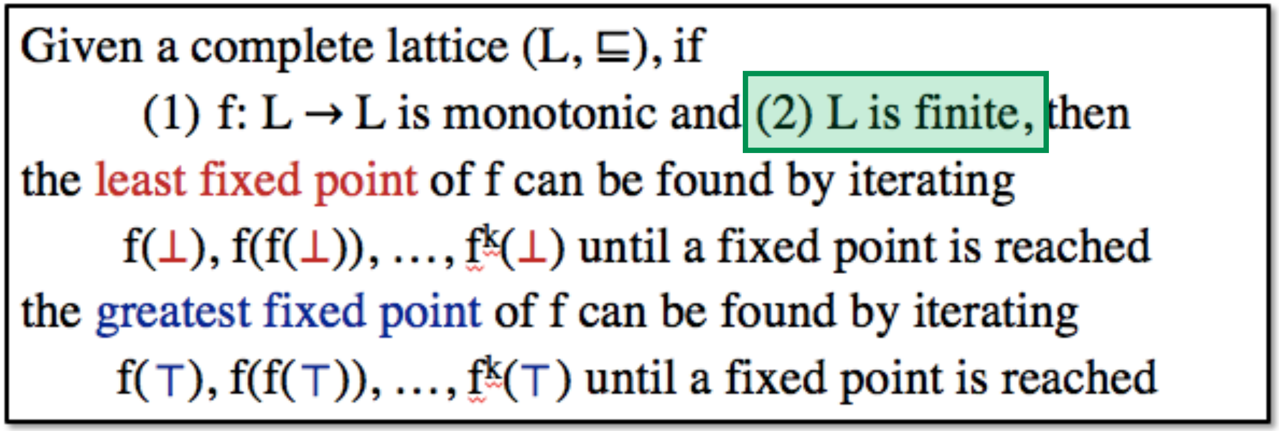
iter 2 $(v_1^2, v_2^2, \ldots, v_k^2)$

$\vdots$

iter i $(v_1^i, v_2^i, \ldots, v_k^i)$

iter i+1 $(v_1^i, v_2^i, \ldots, v_k^i)$

If a product lattice L^k is a product of complete (and finite) lattices, i.e., (L, L, …, L), then L^k is also complete (and finite)



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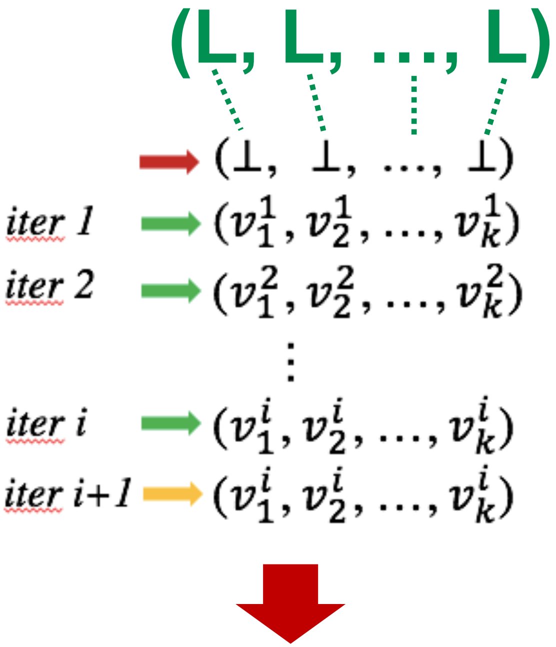
Relate Iterative Algorithm to Fixed-Point Theorem



(L, L, …, L)

$(\perp, \perp, \ldots, \perp)$

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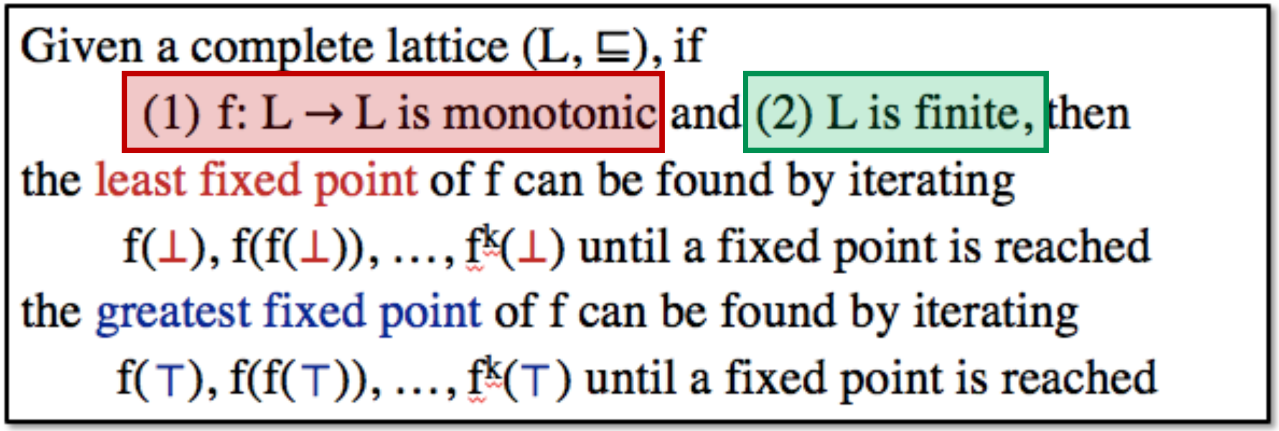
In each iteration, it is equivalent to think that we apply function F which consists of

(1) transfer function f_i: L → L for every node

(2) join/meet function ⊔/⊓: L×L → L for control-flow confluence



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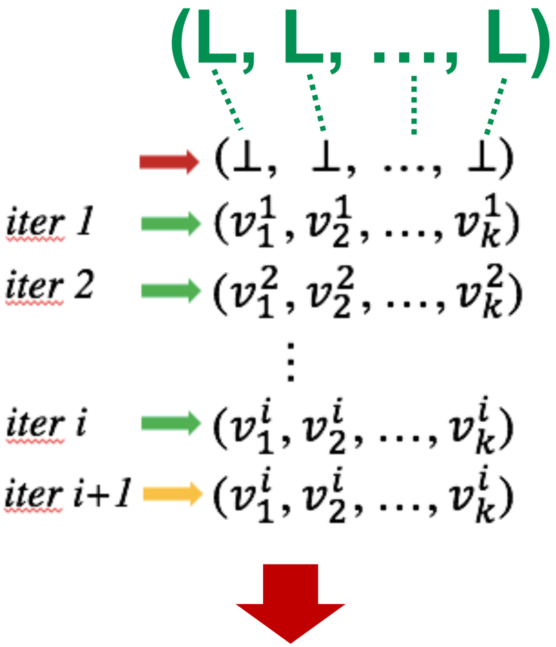
Relate Iterative Algorithm to Fixed-Point Theorem



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⋮

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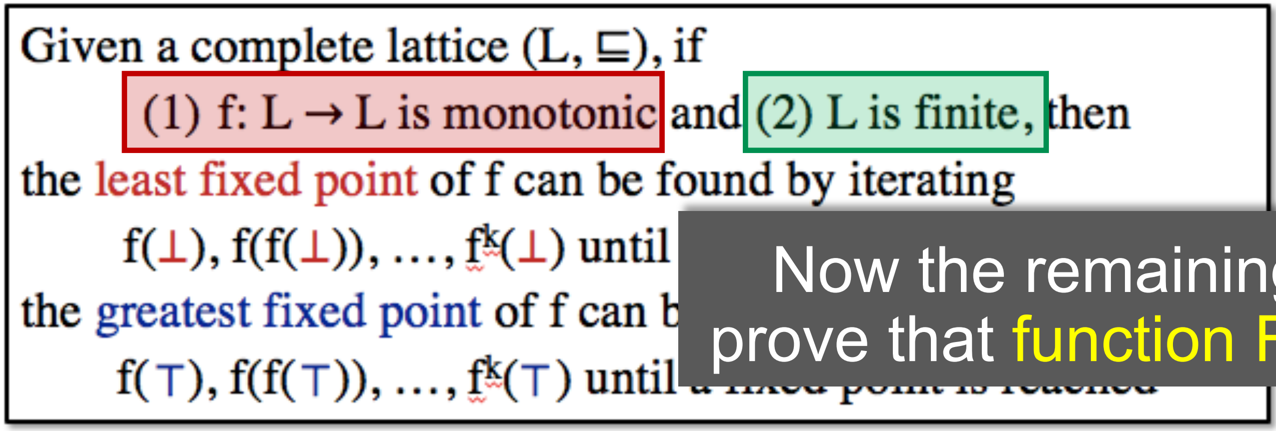

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the greatest fixed point of f can b

$f(\top), f(f(\top)), \ldots, f^k(\top)$ until

Now the remaining issue is to prove that function F is monotonic

Prove Function F is Monotonic

In each iteration, it is equivalent to think that we apply function F which consists of
(1) transfer function f_i: L → L for every node
(2) join/meet function $\sqcup / \sqcap$: $L \times L \rightarrow L$ for control-flow confluence

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Gen/Kill function is monotonic

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Actually the binary operator is a basic case of L × L ×… × L,

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We want to show that $\sqcup$ is monotonic

Proof.

$\forall x, y, z \in L$, $x \sqsubseteq y$, we want to prove $x \sqcup z \sqsubseteq y \sqcup z$

Prove Function F is Monotonic

In each iteration, it is equivalent to think that we apply function F which consists of
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by the definition of $\sqcup$, $y \sqsubseteq y \sqcup z$

by transitivity of $\sqsubseteq$, $x \sqsubseteq y \sqcup z$

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by the definition of $\sqcup$, $y \sqsubseteq y \sqcup z$

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thus $y \sqcup z$ is an upper bound for x, and also for z (by $\sqcup$'s definition)

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Gen/Kill function is monotonic

We want to show that ⊔ is monotonic

Proof.

∀x, y, z ∈ L, x ⊑ y, we want to prove x ⊔ z ⊑ y ⊔ z

by the definition of ⊔, y ⊑ y ⊔ z

by transitivity of ⊑, x ⊑ y ⊔ z

thus y ⊔ z is an upper bound for x, and also for z (by ⊔'s definition)

as x ⊔ z is the least upper bound of x and z

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(1) transfer function f_i: L → L for every node

(2) join/meet function $\sqcup / \sqcap$: $L \times L \rightarrow L$ for control-flow confluence

Actually the binary operator is a basic case of $L \times L \times \ldots \times L$,

Gen/Kill function is monotonic

We want to show that $\sqcup$ is monotonic

Proof.

$\forall x, y, z \in L$, $x \sqsubseteq y$, we want to prove $x \sqcup z \sqsubseteq y \sqcup z$

by the definition of $\sqcup$, $y \sqsubseteq y \sqcup z$

by transitivity of $\sqsubseteq$, $x \sqsubseteq y \sqcup$

thus $y \sqcup z$ is an upp ⊔´s definition)

as $x \sqcup z$ is the least ound of x and z

thus $x \sqcup z \sqsubseteq y \sqcup z$

Thus the fixed point theorem applies to the iterative algorithm for data flow analysis

Review The Questions We Have Seen Before

The iterative algorithm (or the IN/OUT equation system) produces a solution to a data flow analysis

- ? Is the algorithm guaranteed to terminate or reach the fixed point, or does it always have a solution?
- ? If so, is there only one solution or only one fixed point? If more than one, is our solution the best one (m

greatest or least fixed point

- When will the algorithm reach the fixed point, or when can we get the solution?

Now what we have just seen is the property (fixed point theorem) for the function on a lattice. ~~We cannot say our iterative algorithm also has that property unless~~ we can relate the algorithm to the fixed point theorem, ~~if possible~~

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When Will the Algorithm Reach the Fixed Point?

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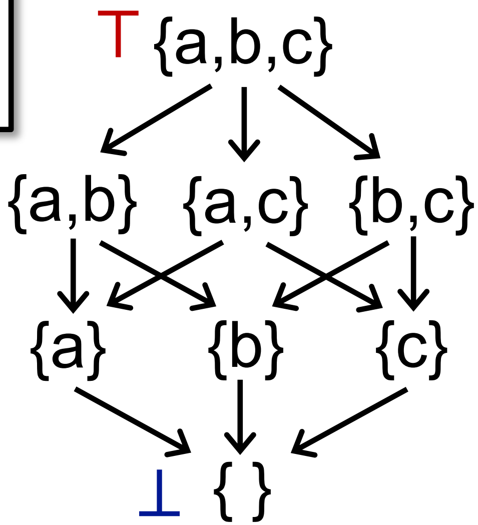
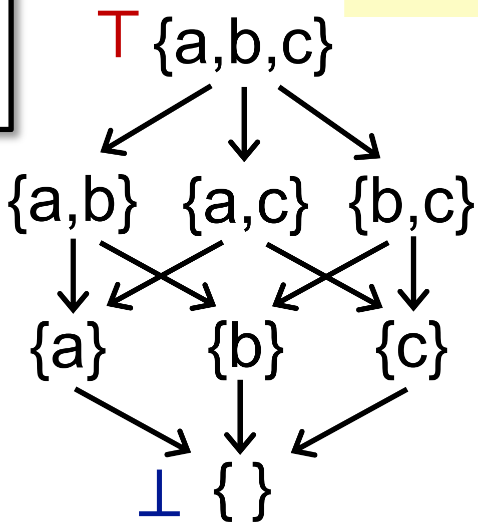
The height of a lattice *h* is the length of the longest path from Top to Bottom in the lattice.



When Will the Algorithm Reach the Fixed Point?

The height of a lattice h is the length of the longest path from Top to Bottom in the lattice.

$h = 3$



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The maximum iterations i needed to reach the fixed point

→ $(\bot, \ \bot, \ \ldots, \ \bot)$

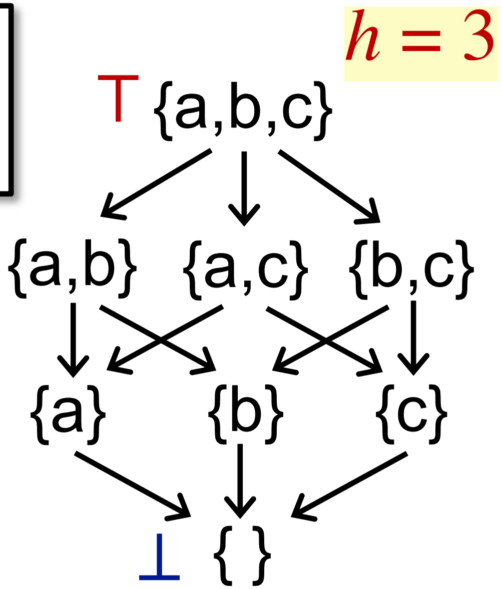
iter 1 → $(v_1^1, v_2^1, \ldots, v_k^1)$

iter 2 → $(v_1^2, v_2^2, \ldots, v_k^2)$

$\vdots$

iter i → $(v_1^i, v_2^i, \ldots, v_k^i)$

iter i+1 → $(v_1^i, v_2^i, \ldots, v_k^i)$



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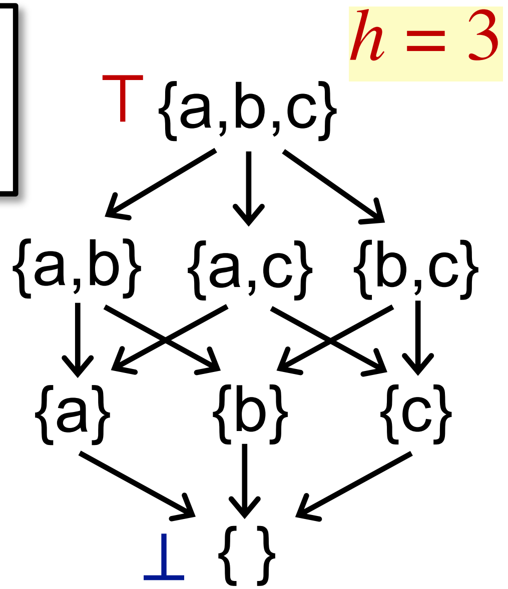

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⋮

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In each iteration, assume only one step in the lattice (upwards or downwards) is made in one node (e.g., one 0->1 in RD)

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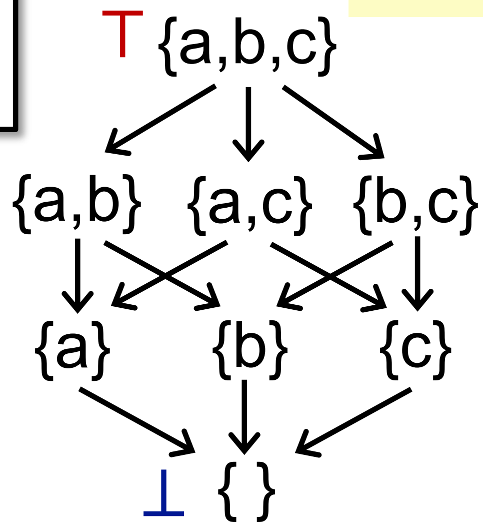

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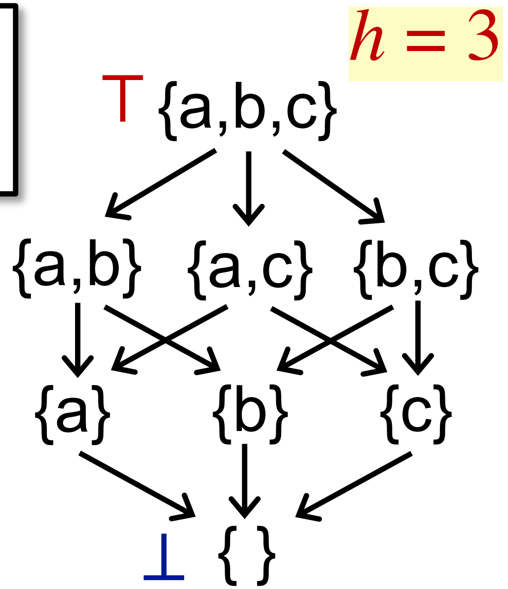

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In each iteration, assume only one step in the lattice (upwards or downwards) is made in one node (e.g., one 0->1 in RD)

Assume the lattice height is h and the number of nodes in CFG is k

We need at most $i = h*k$ iterations

Review The Questions We Have Seen Before

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YES

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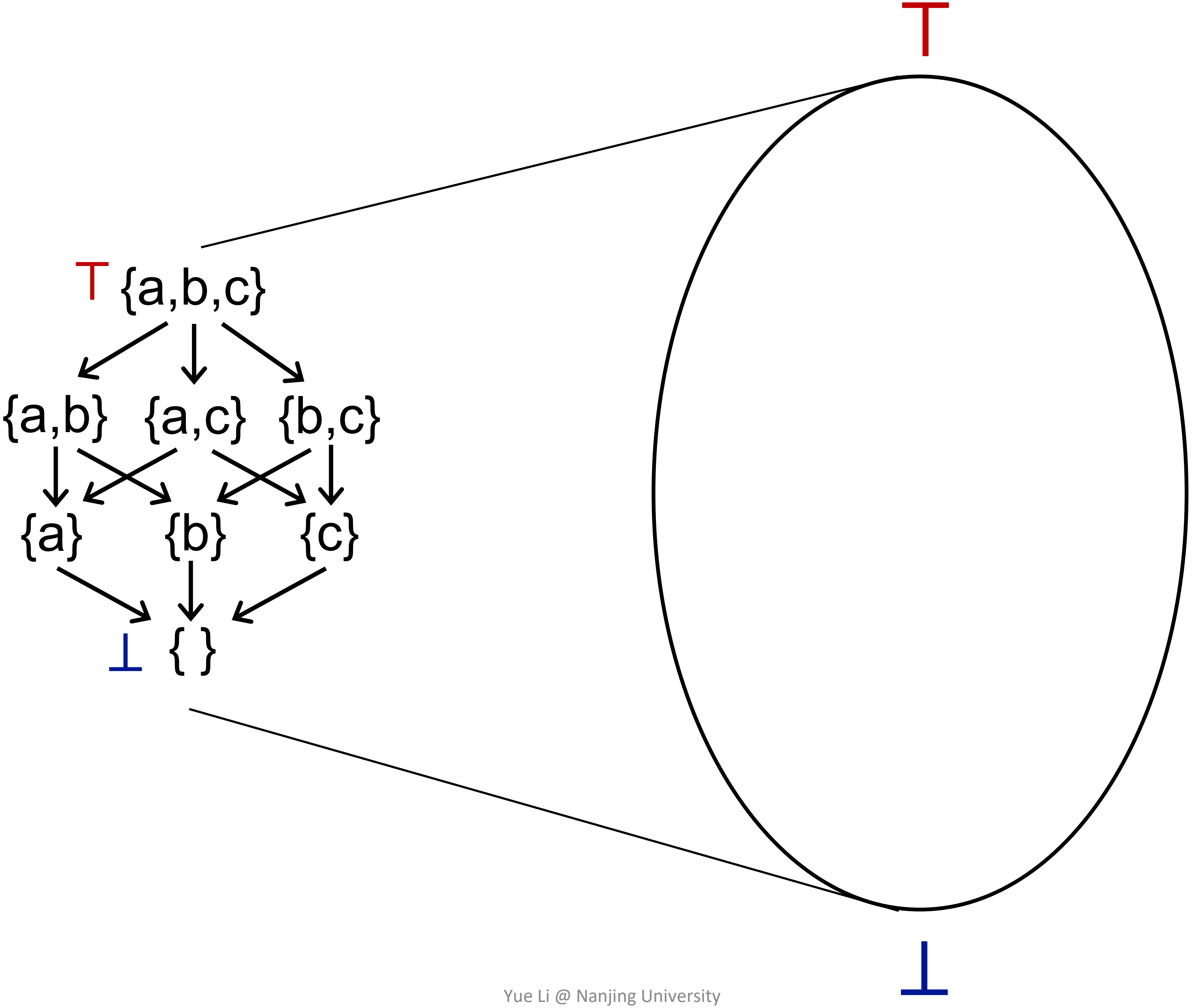
Worst case of #iterations:
the product of the lattice height and the number of nodes in CFG

May and Must Analyses, a Lattice View



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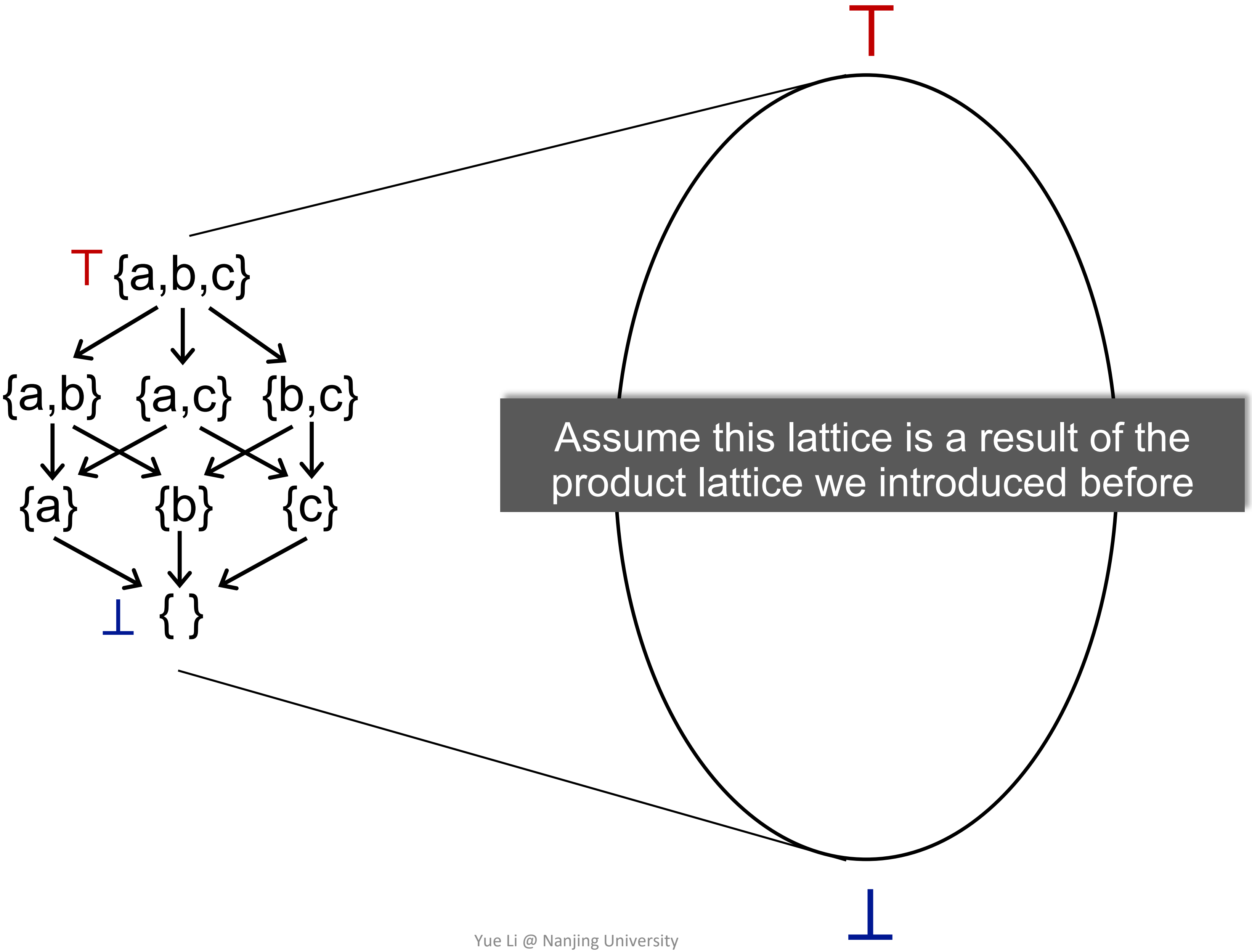
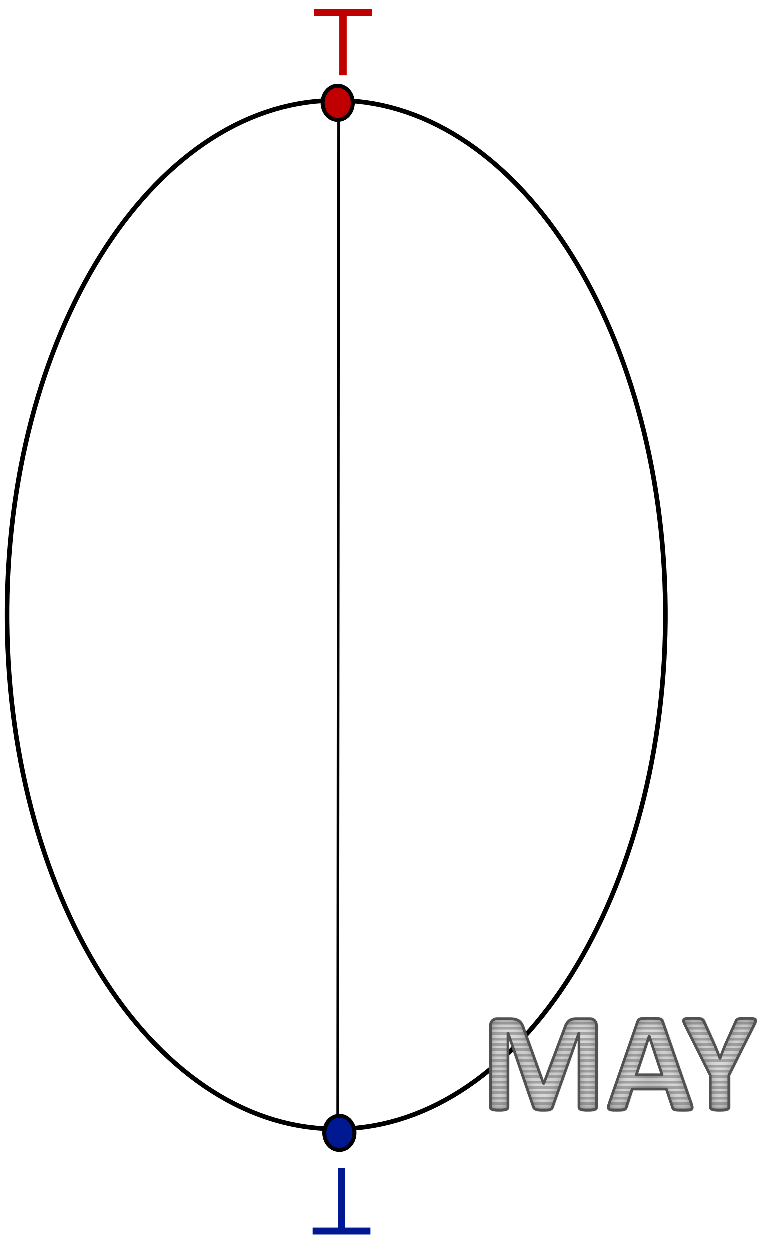


Assume this lattice is a result of the product lattice we introduced before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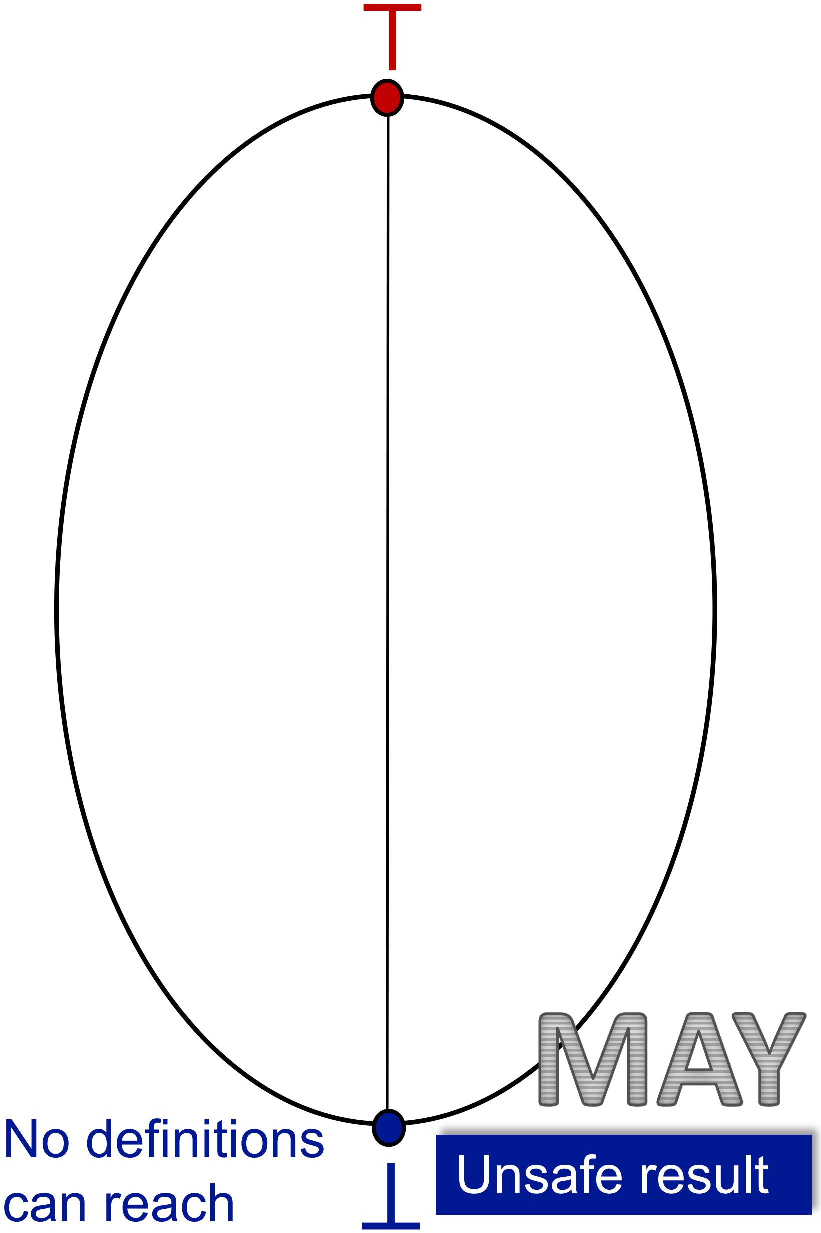
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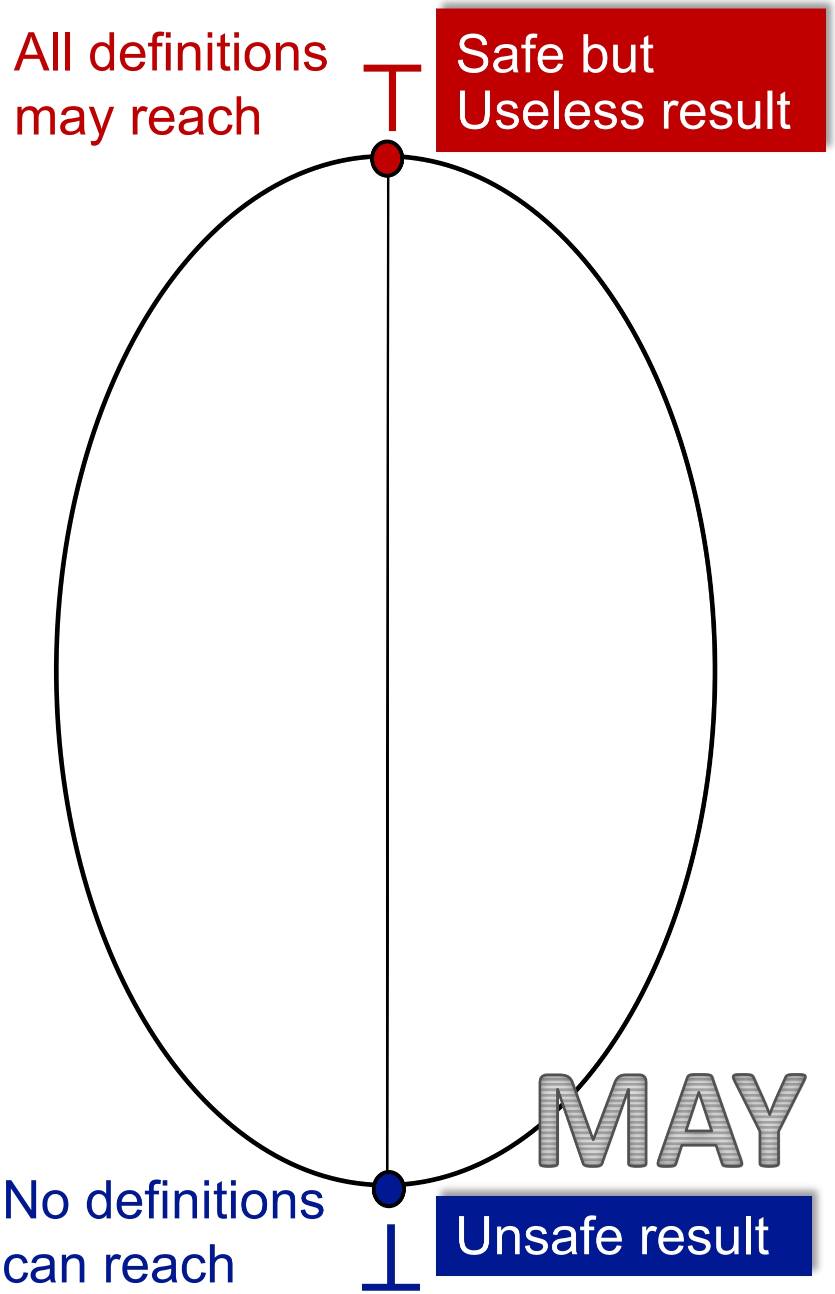

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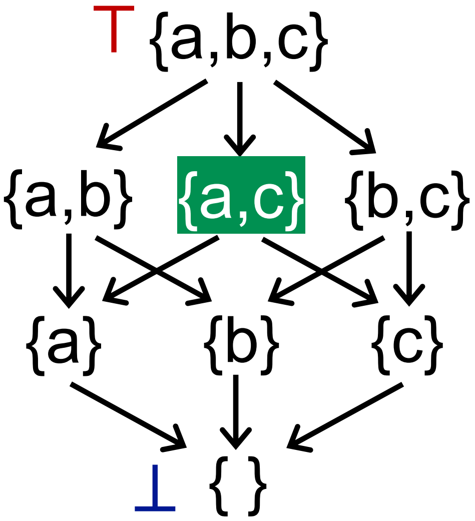

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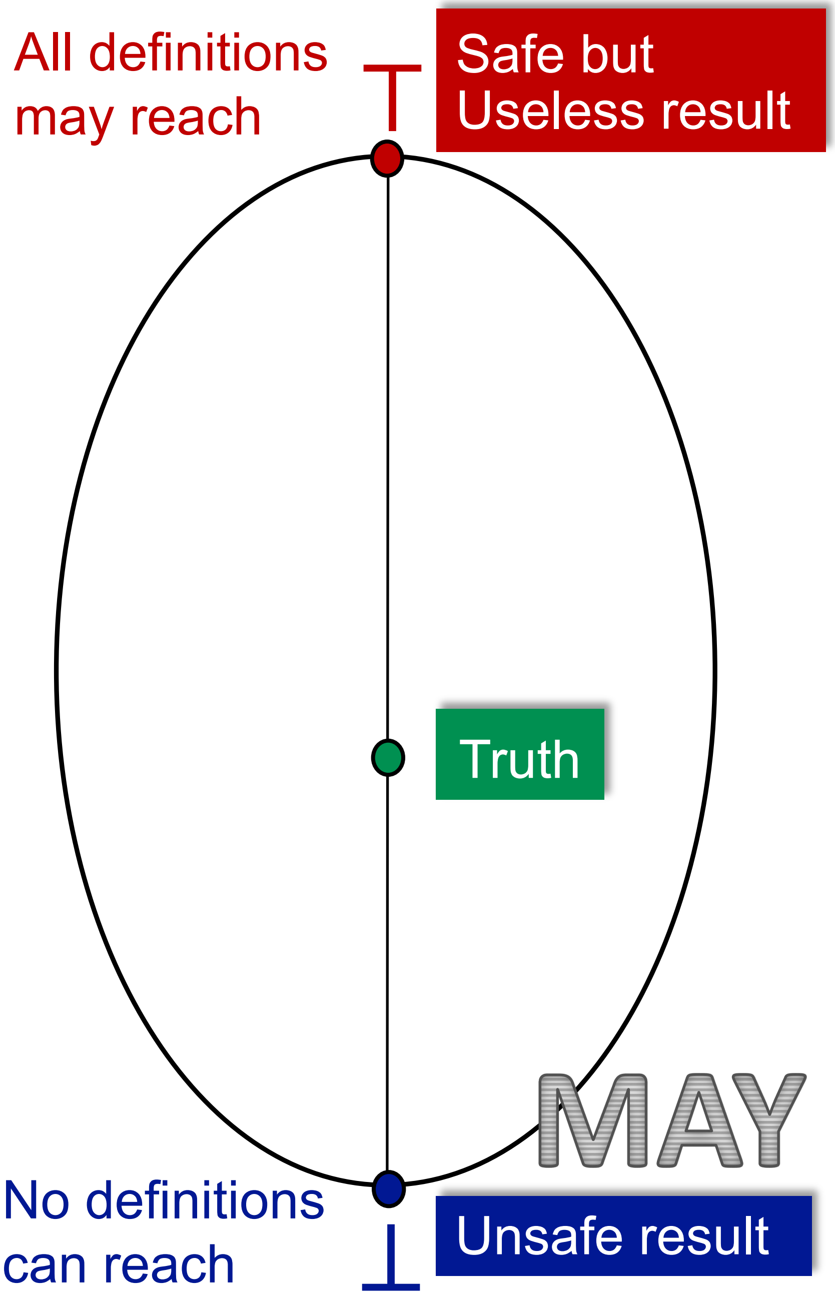

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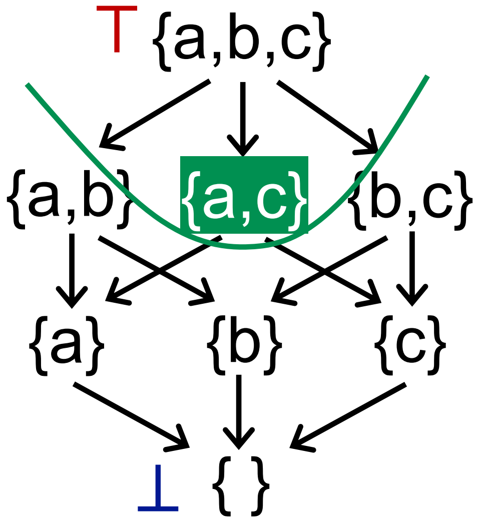
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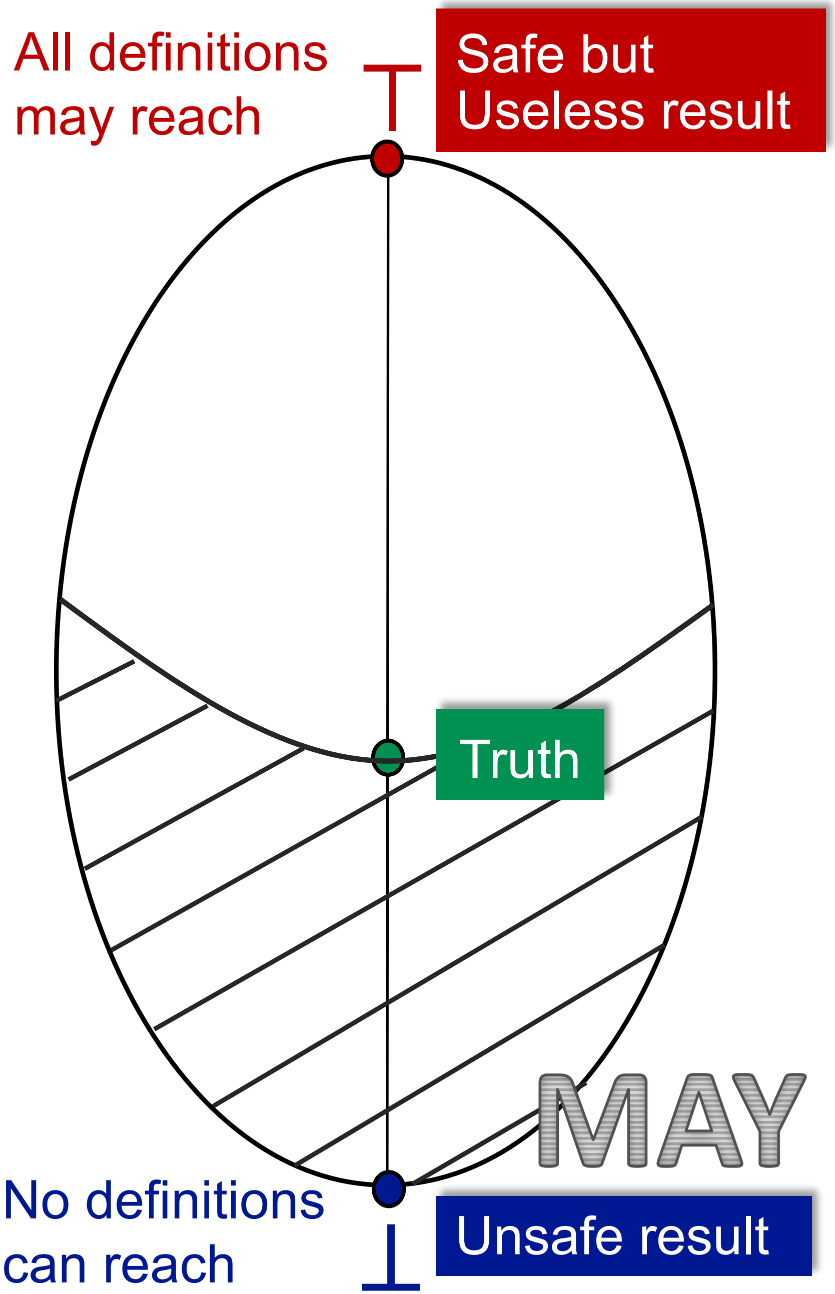
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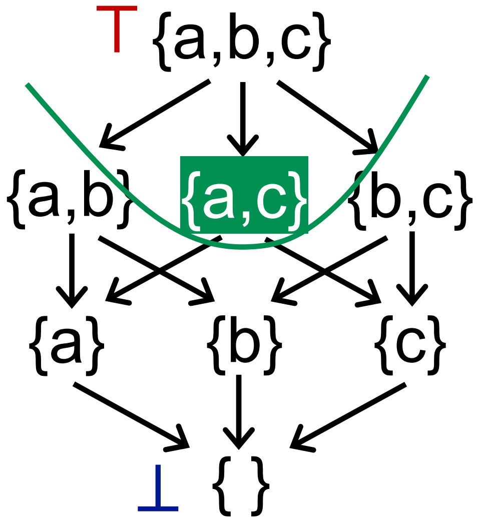

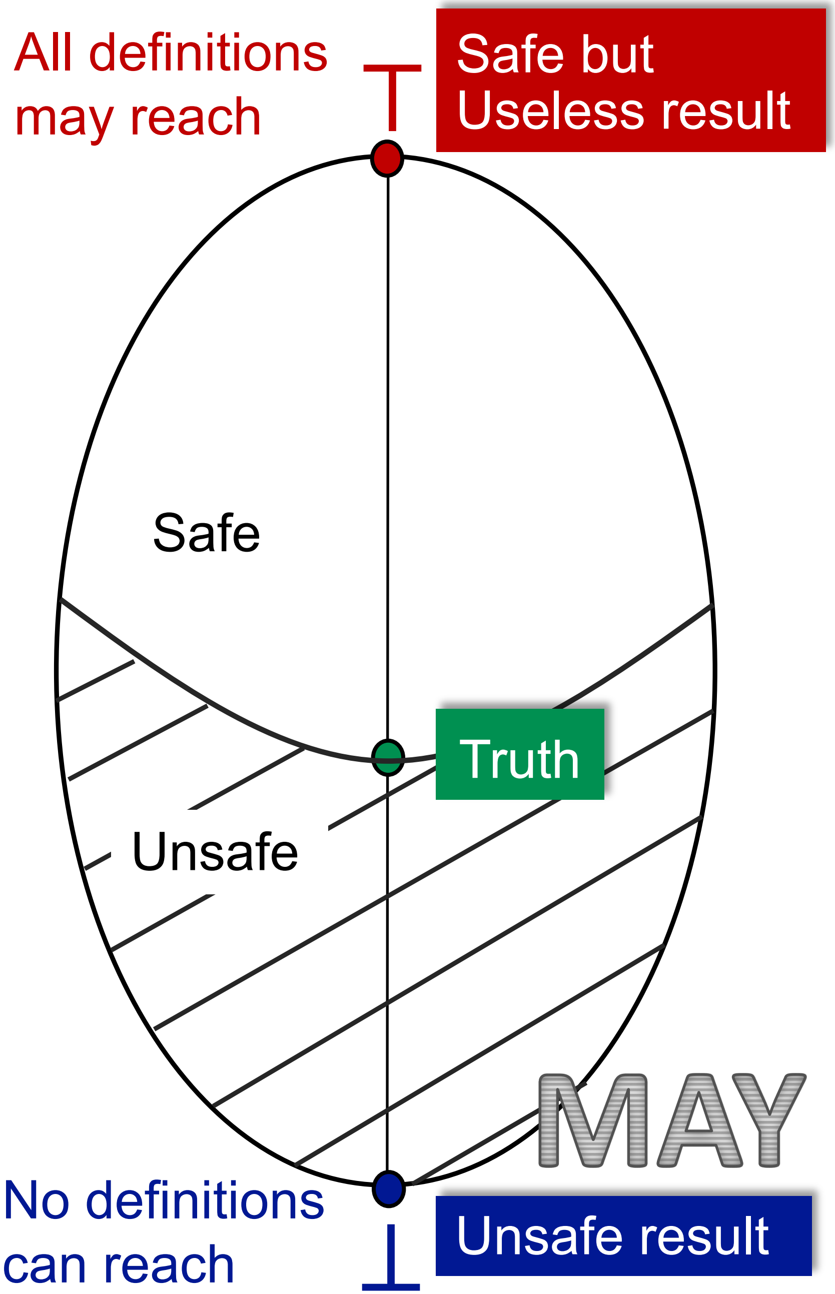
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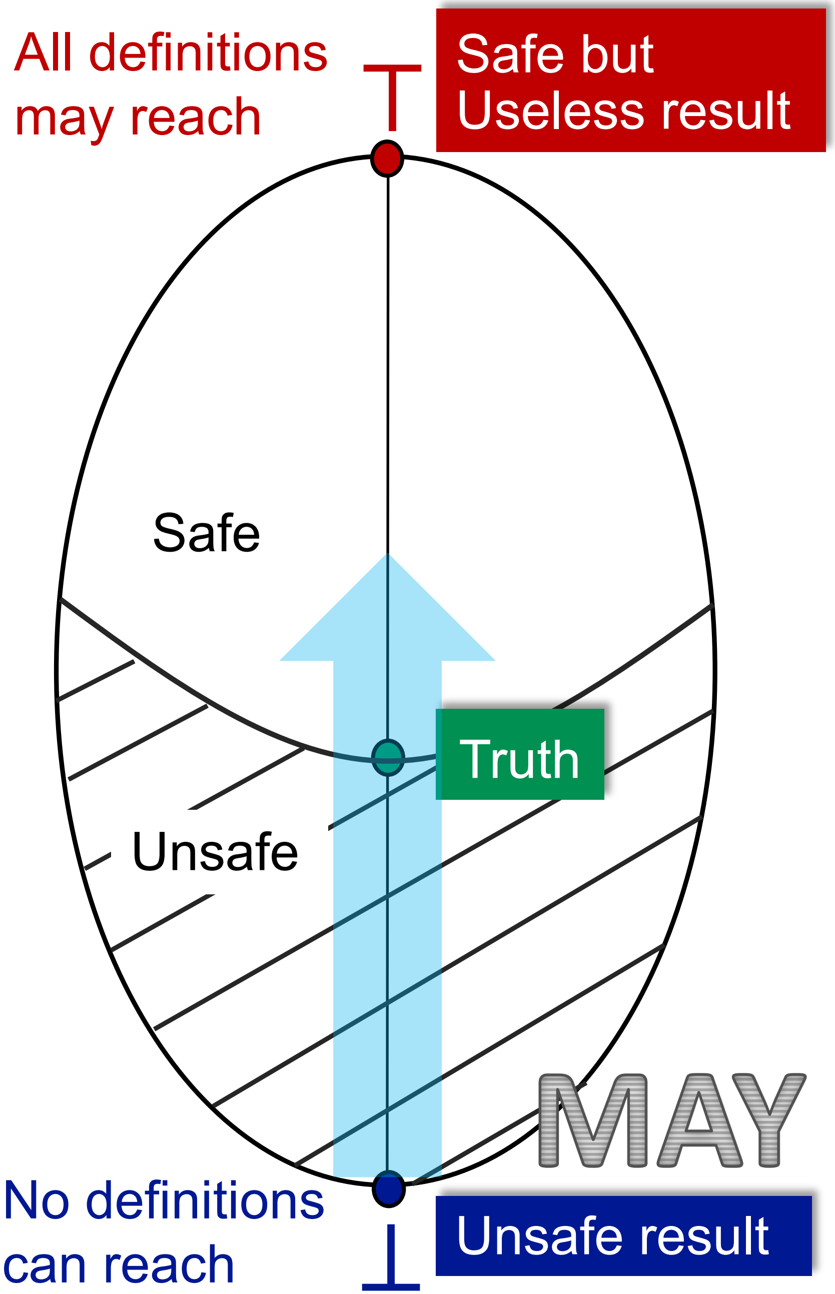
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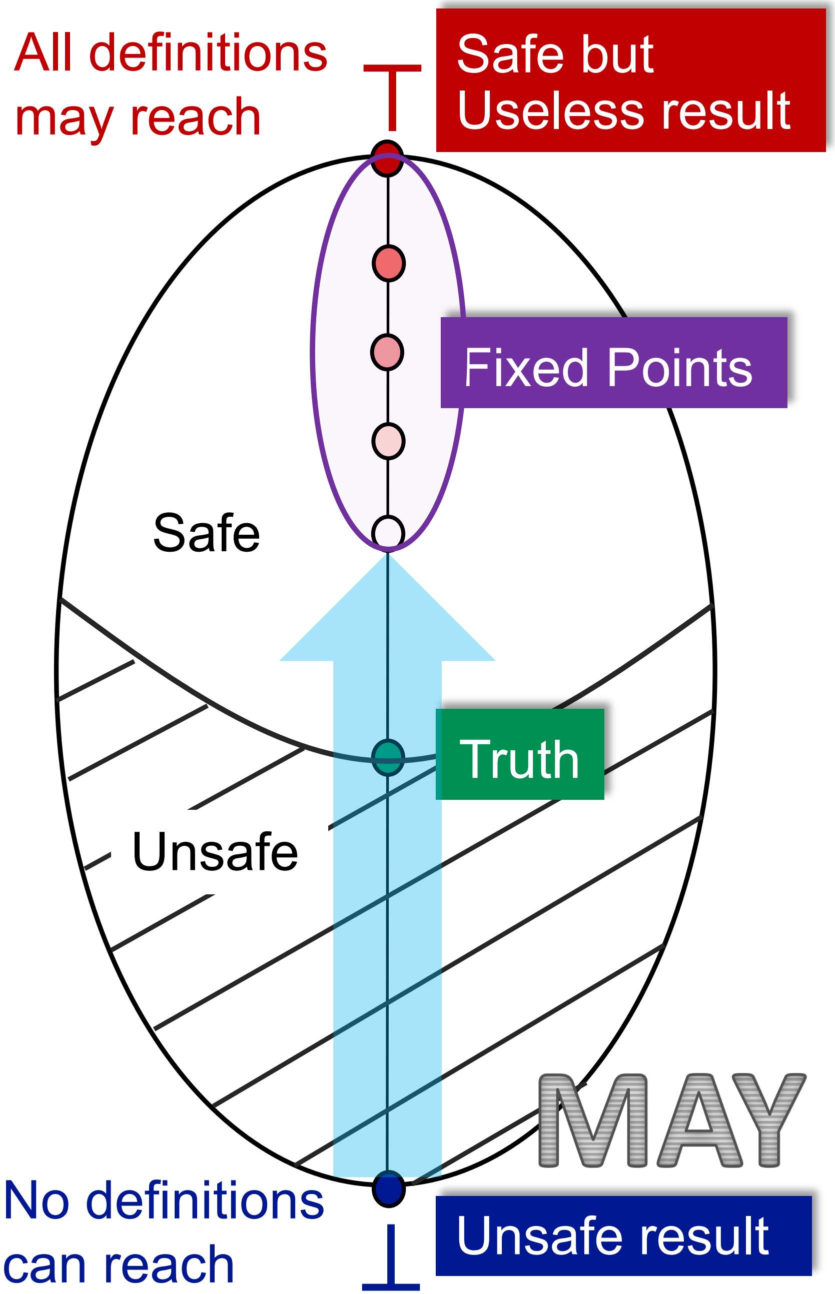
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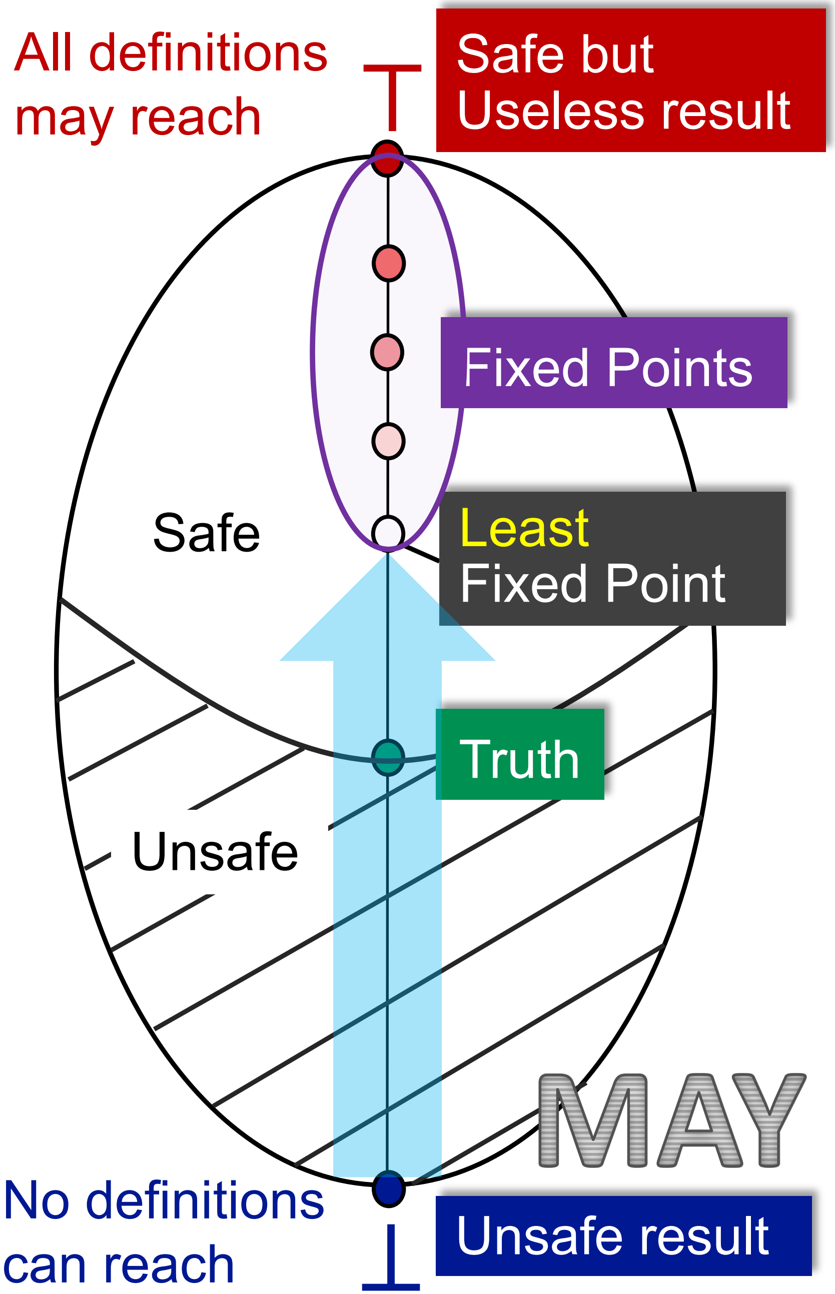
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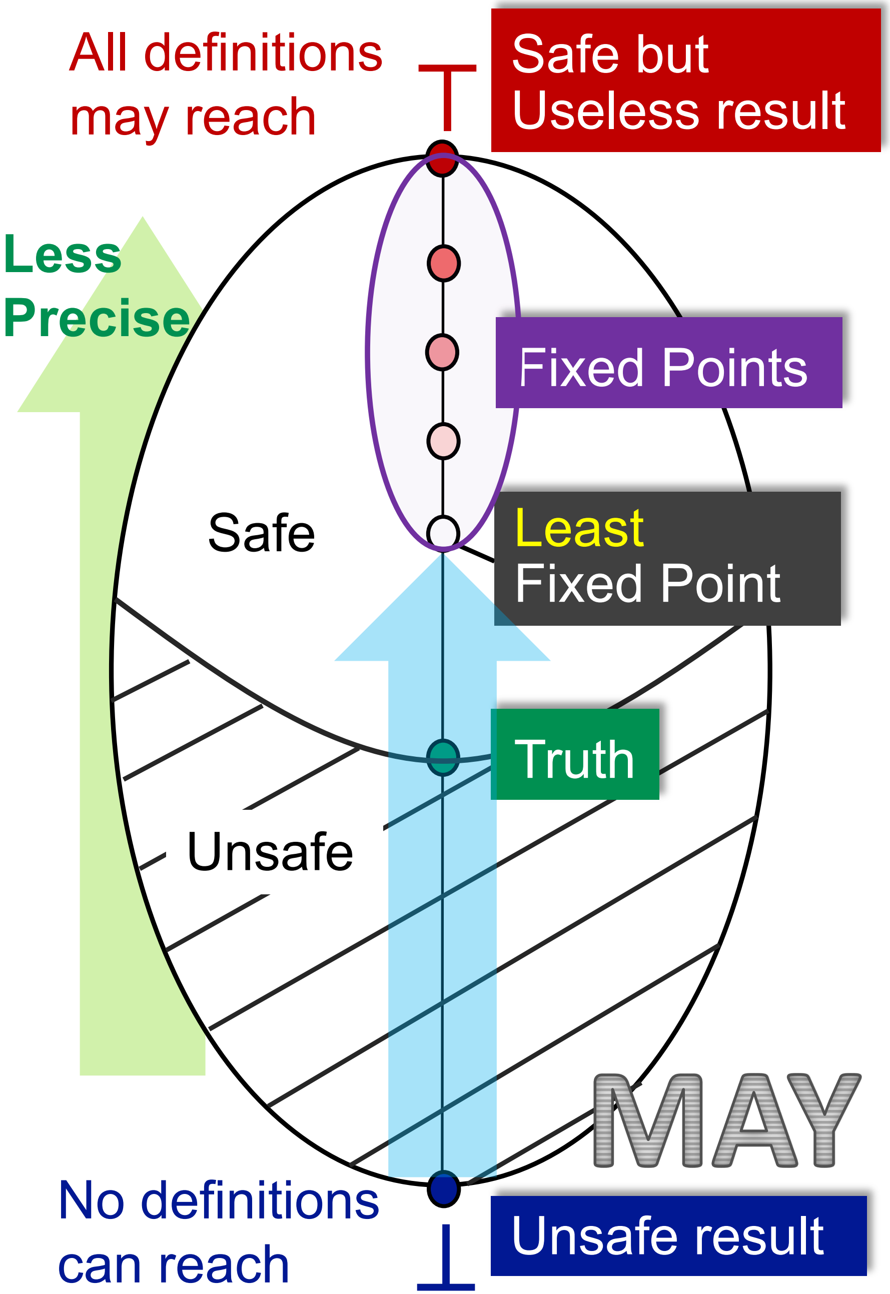
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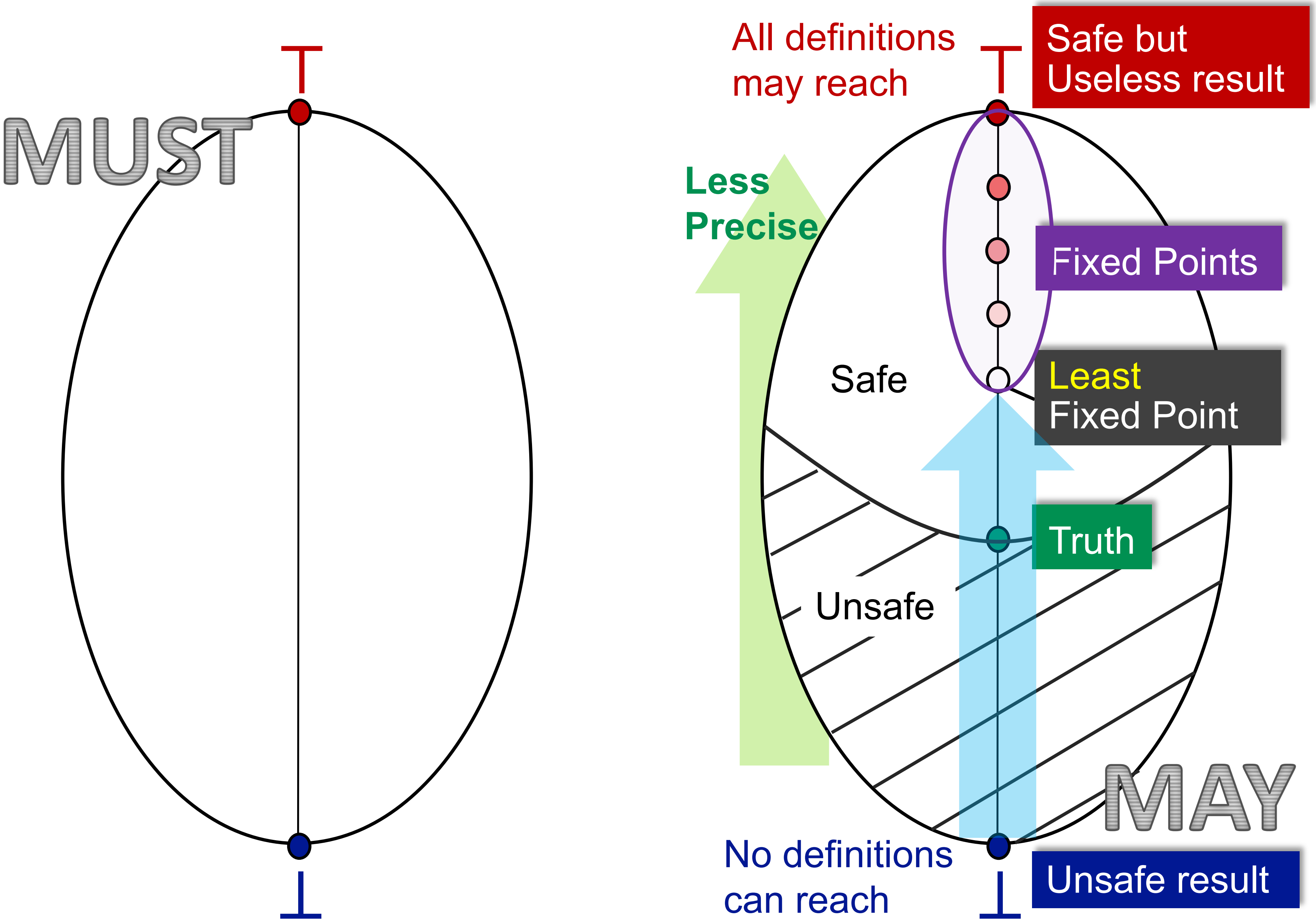

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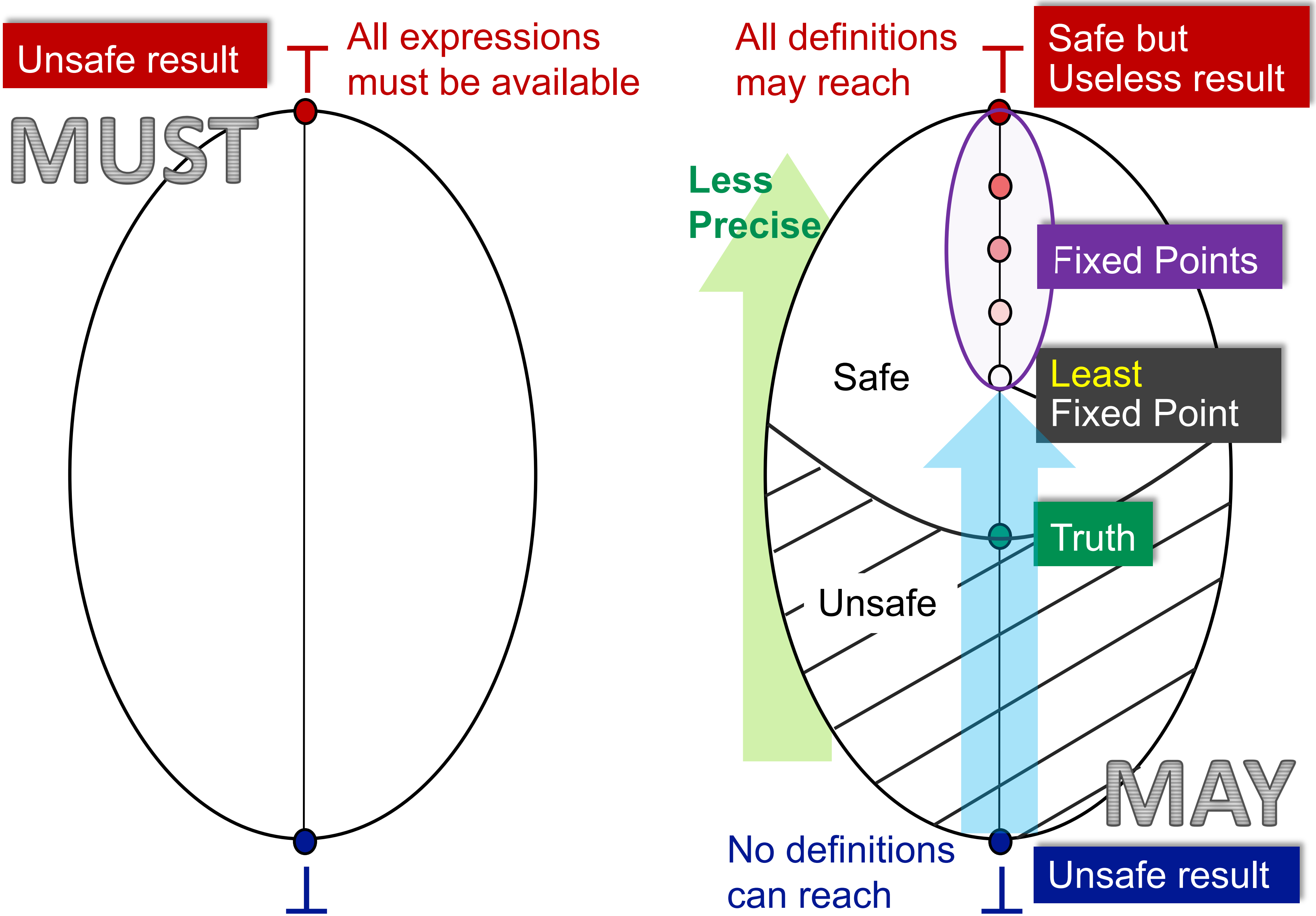
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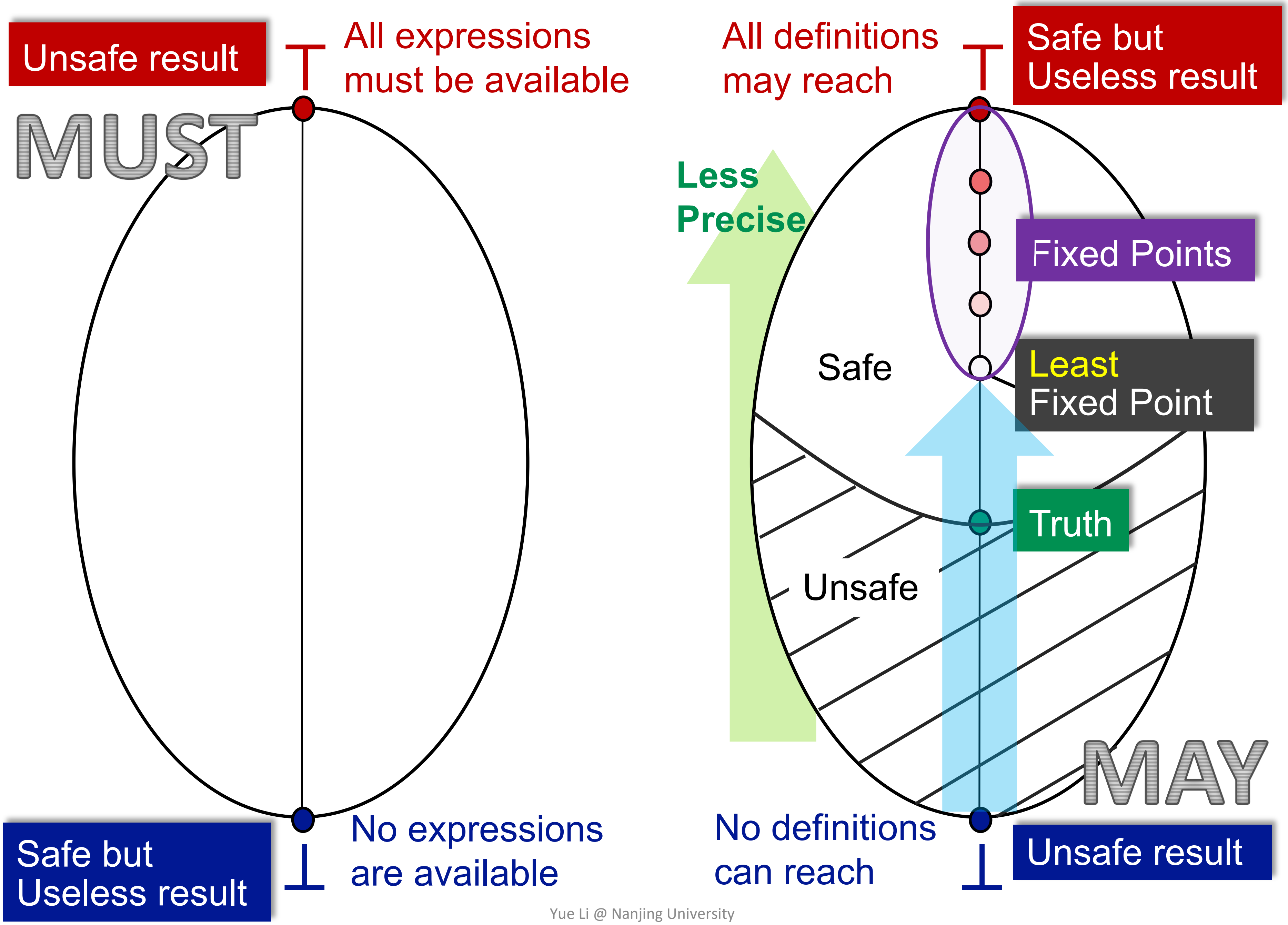
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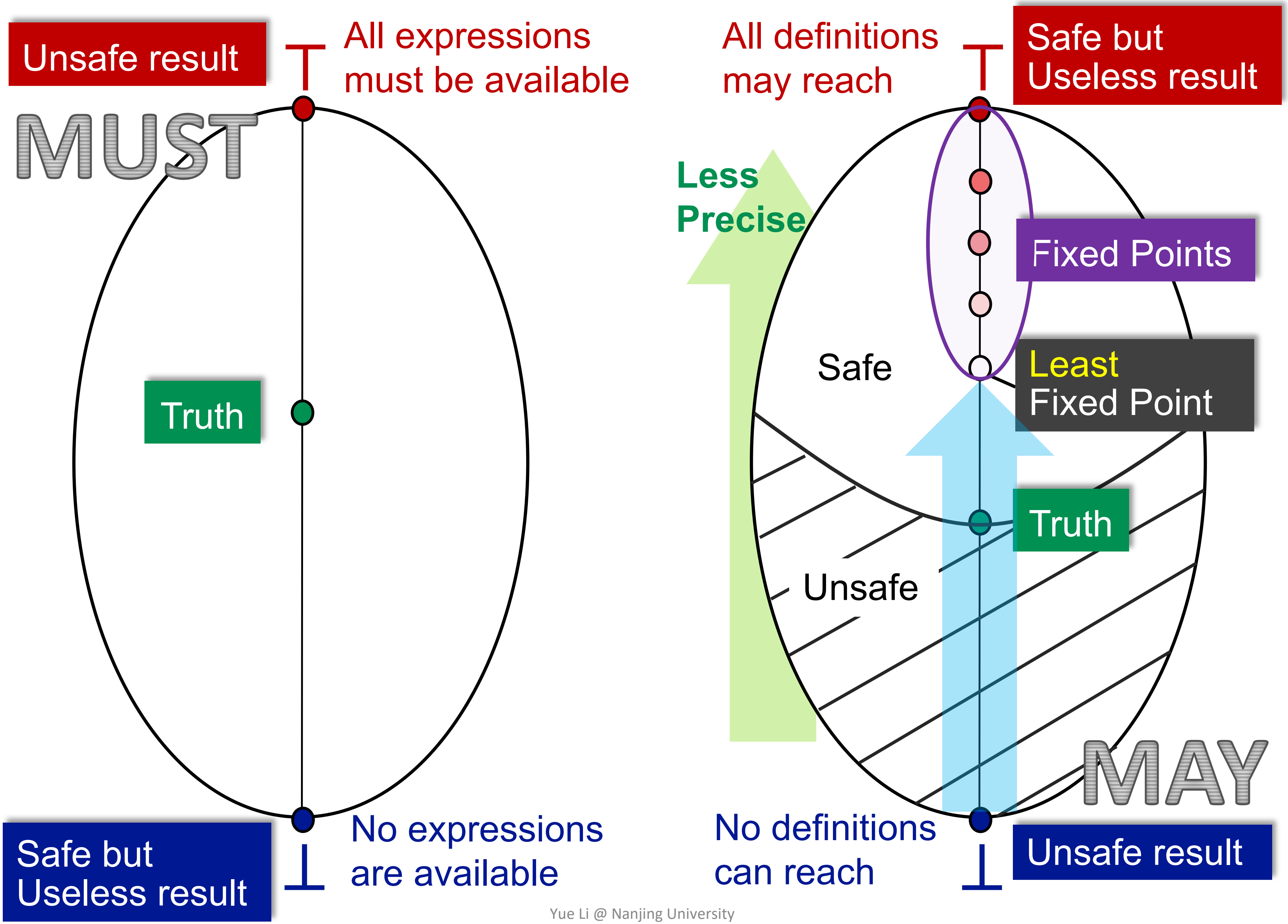

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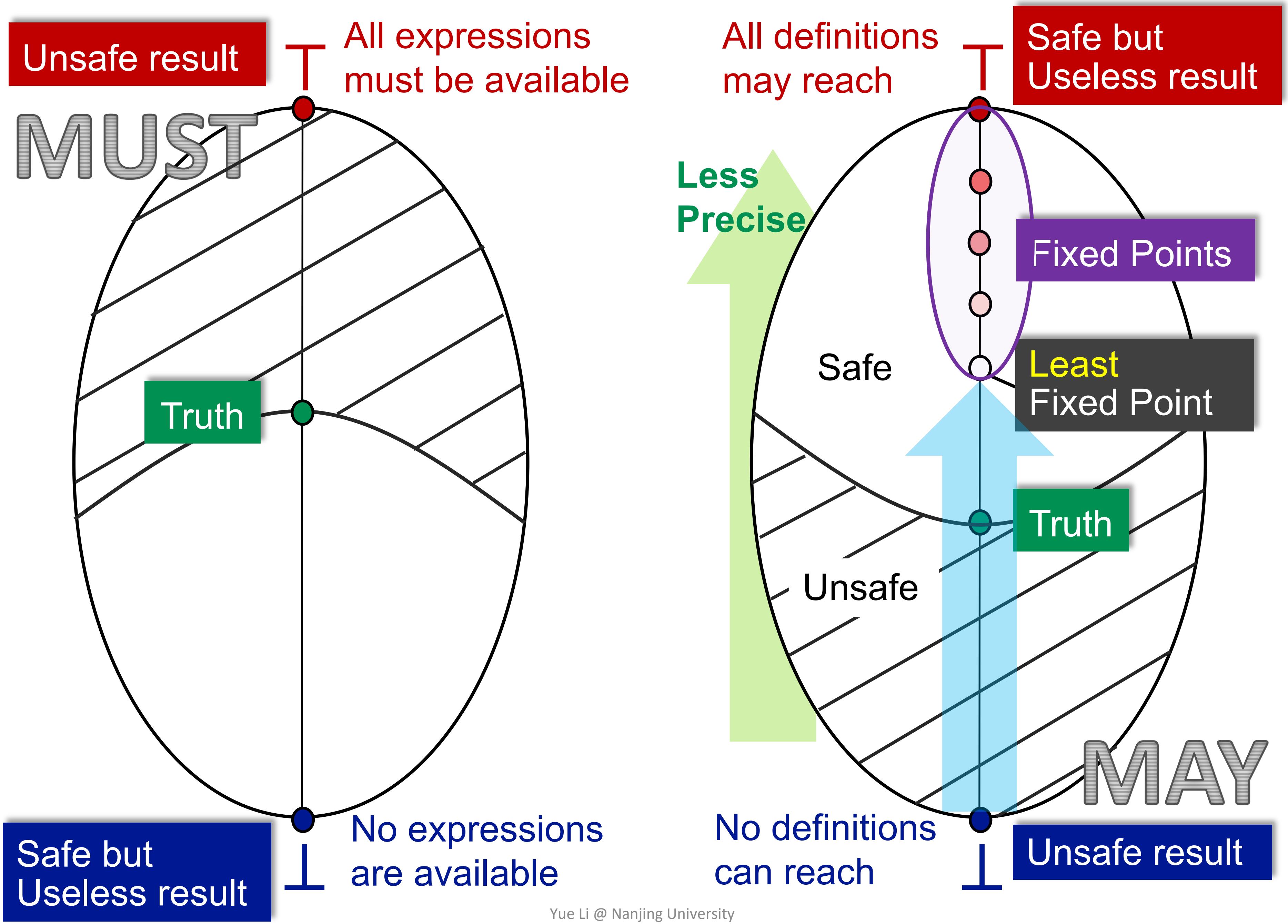

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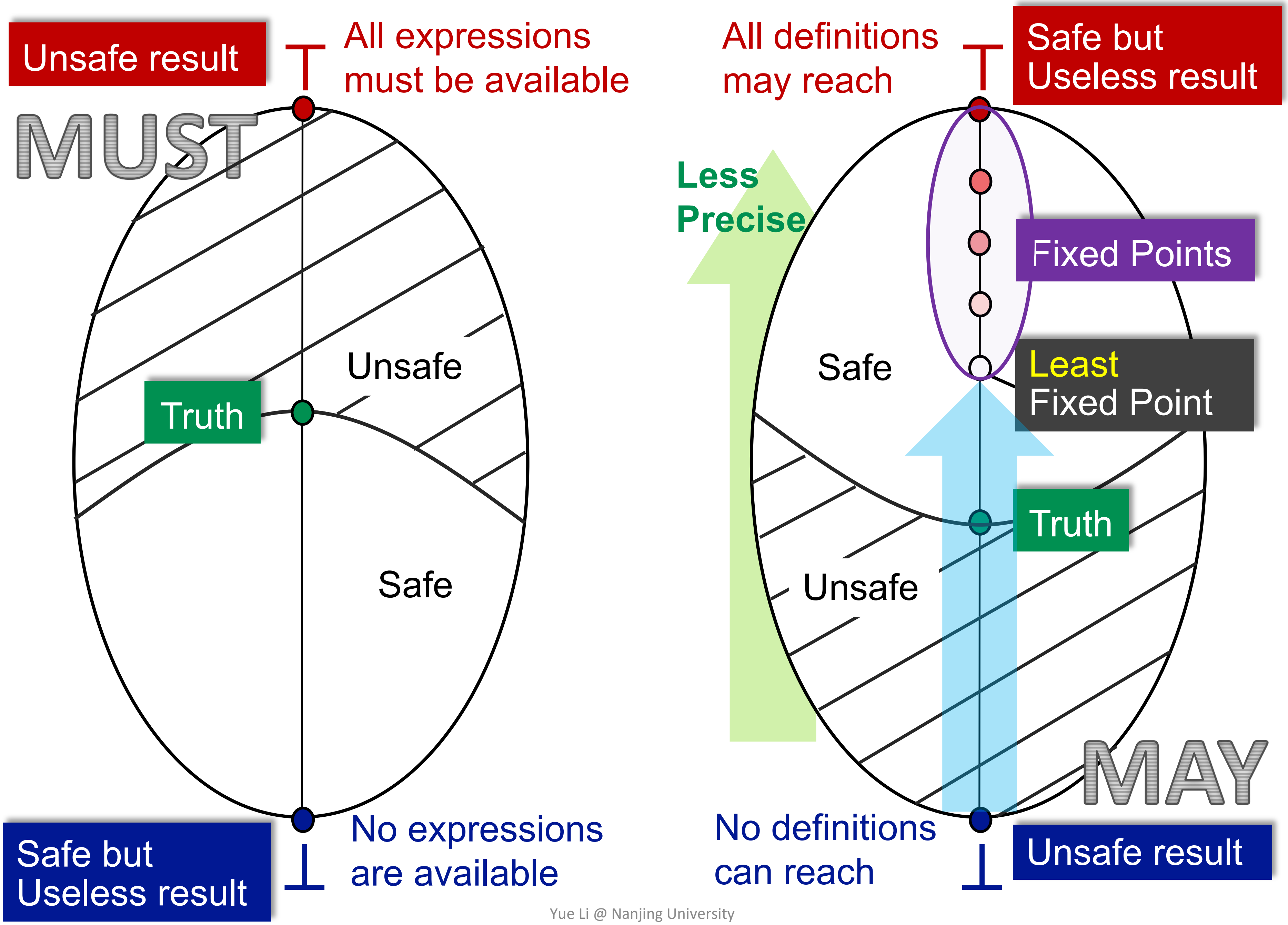

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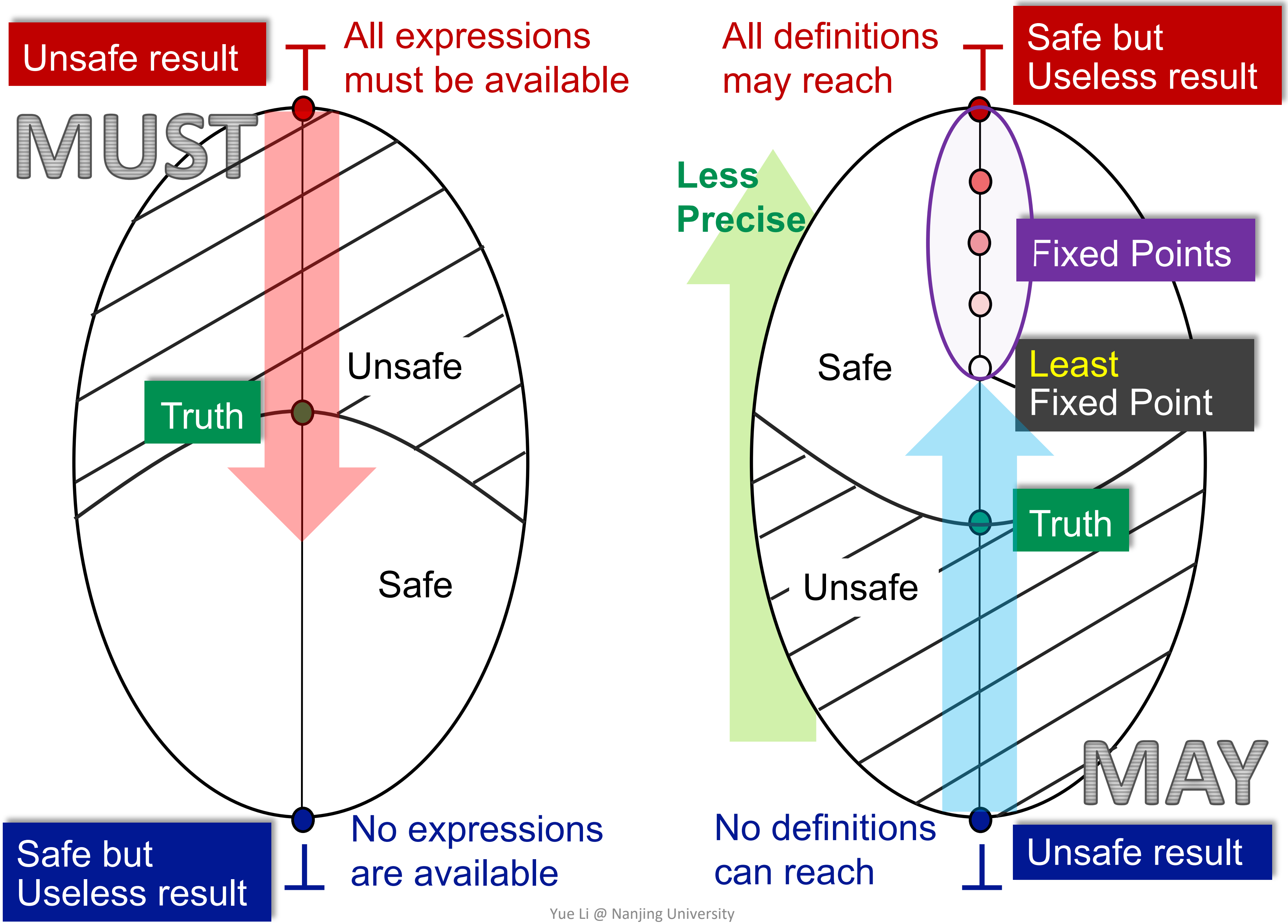

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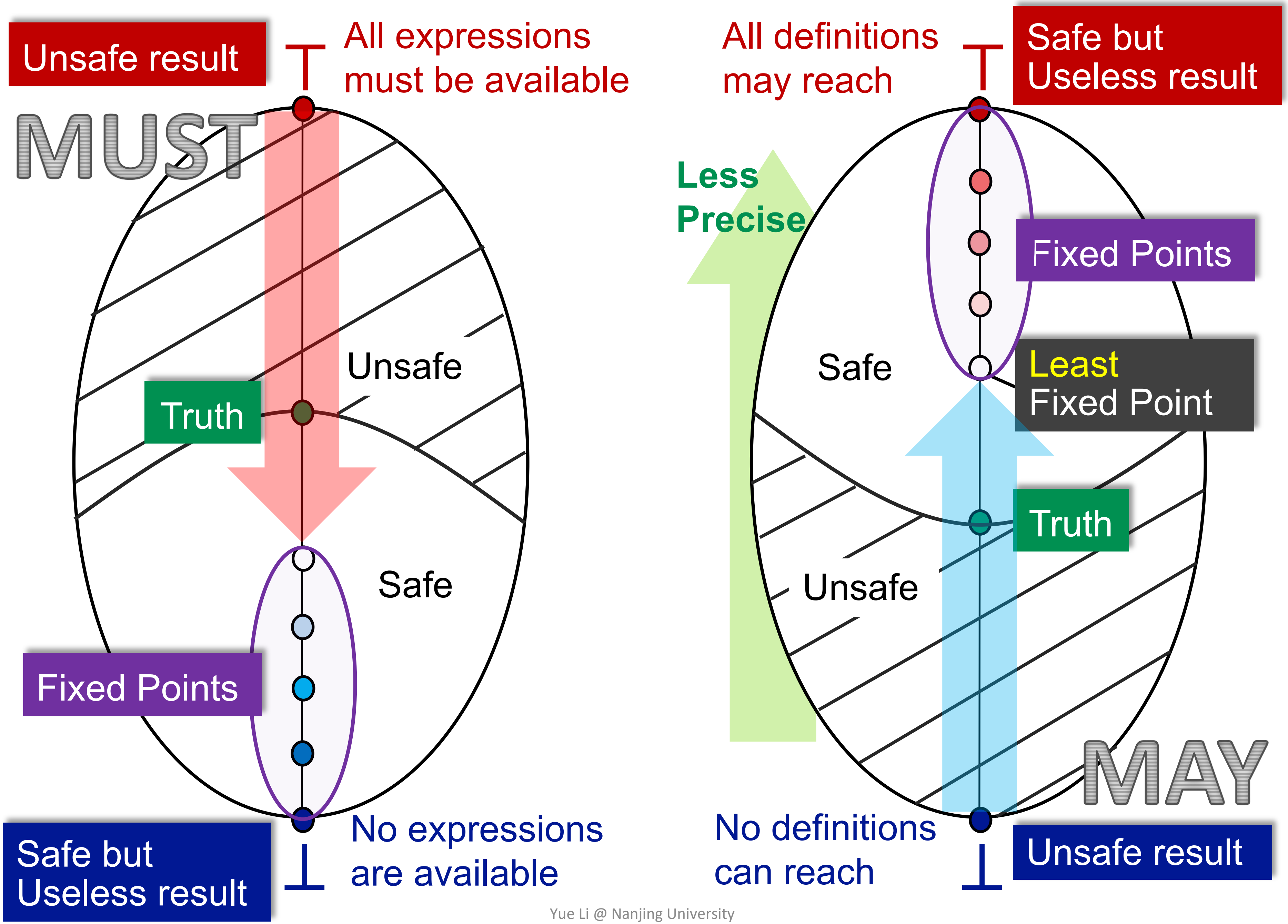

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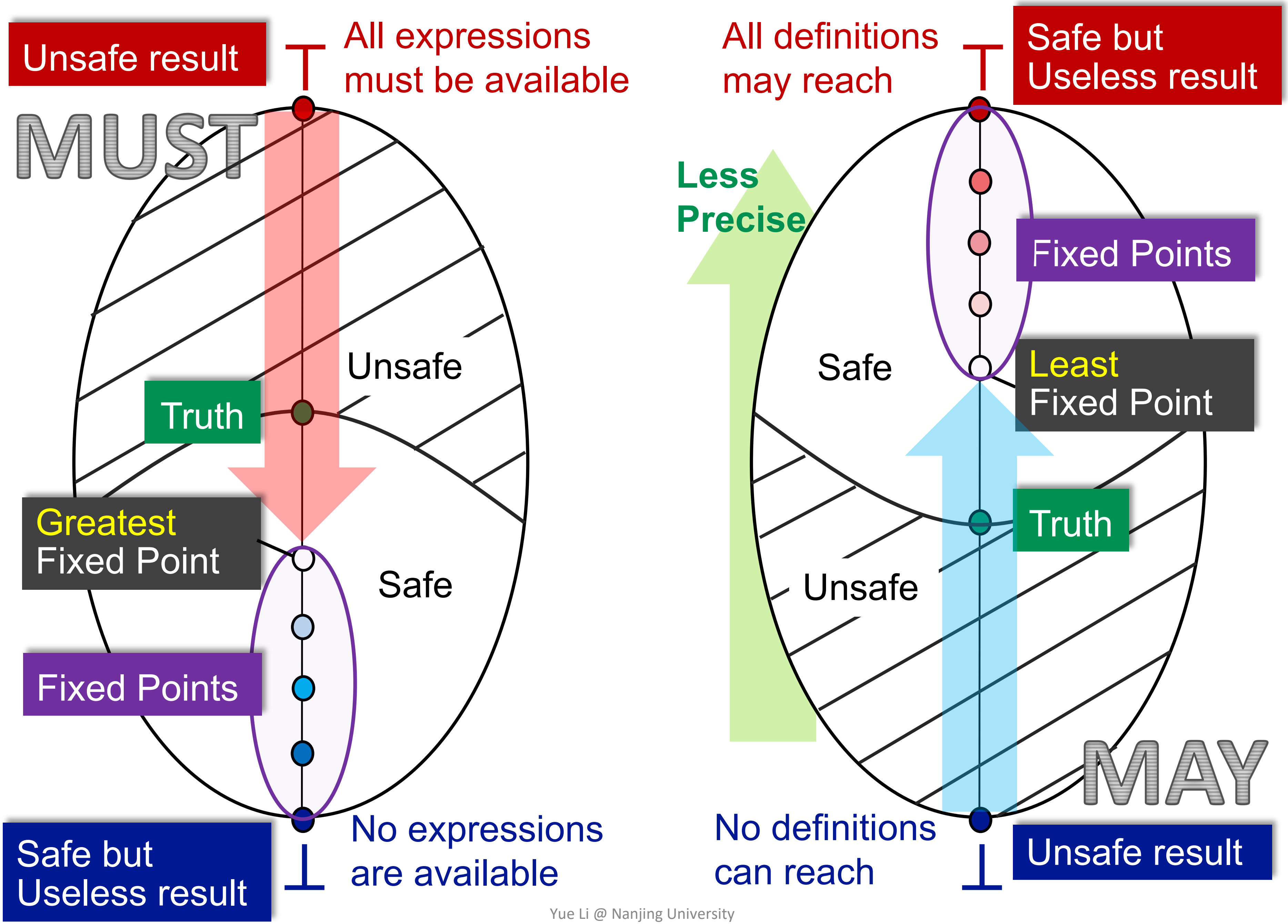
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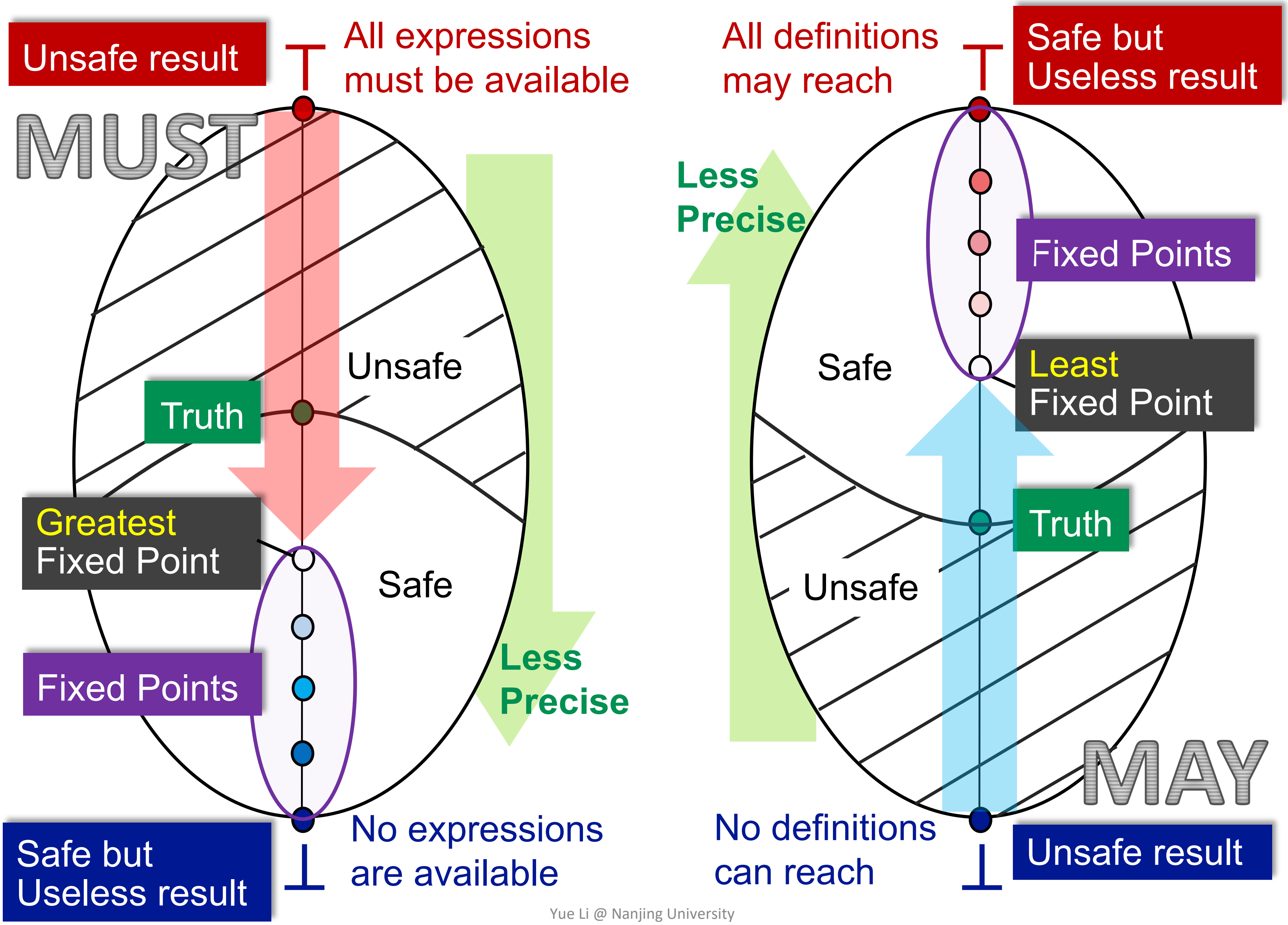

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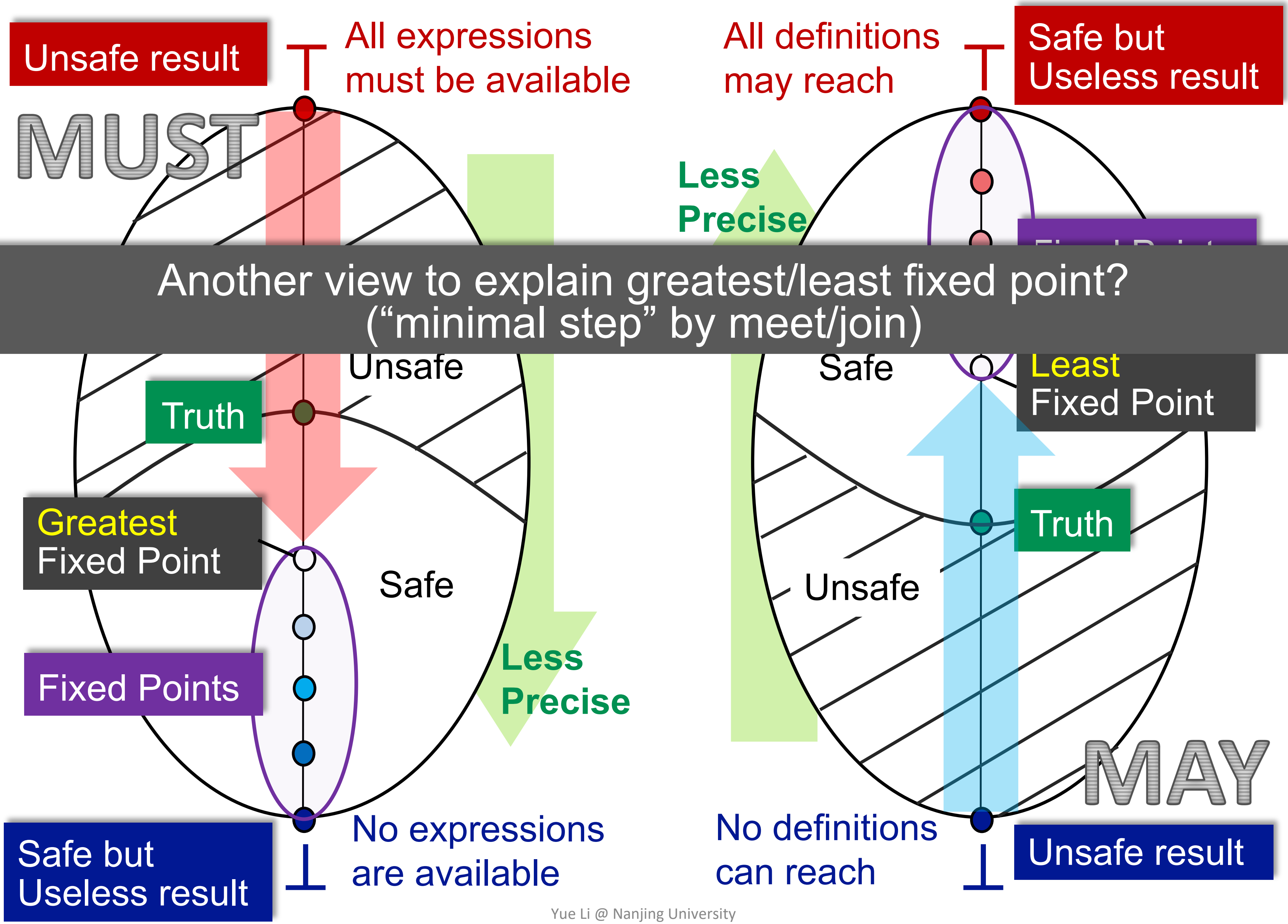
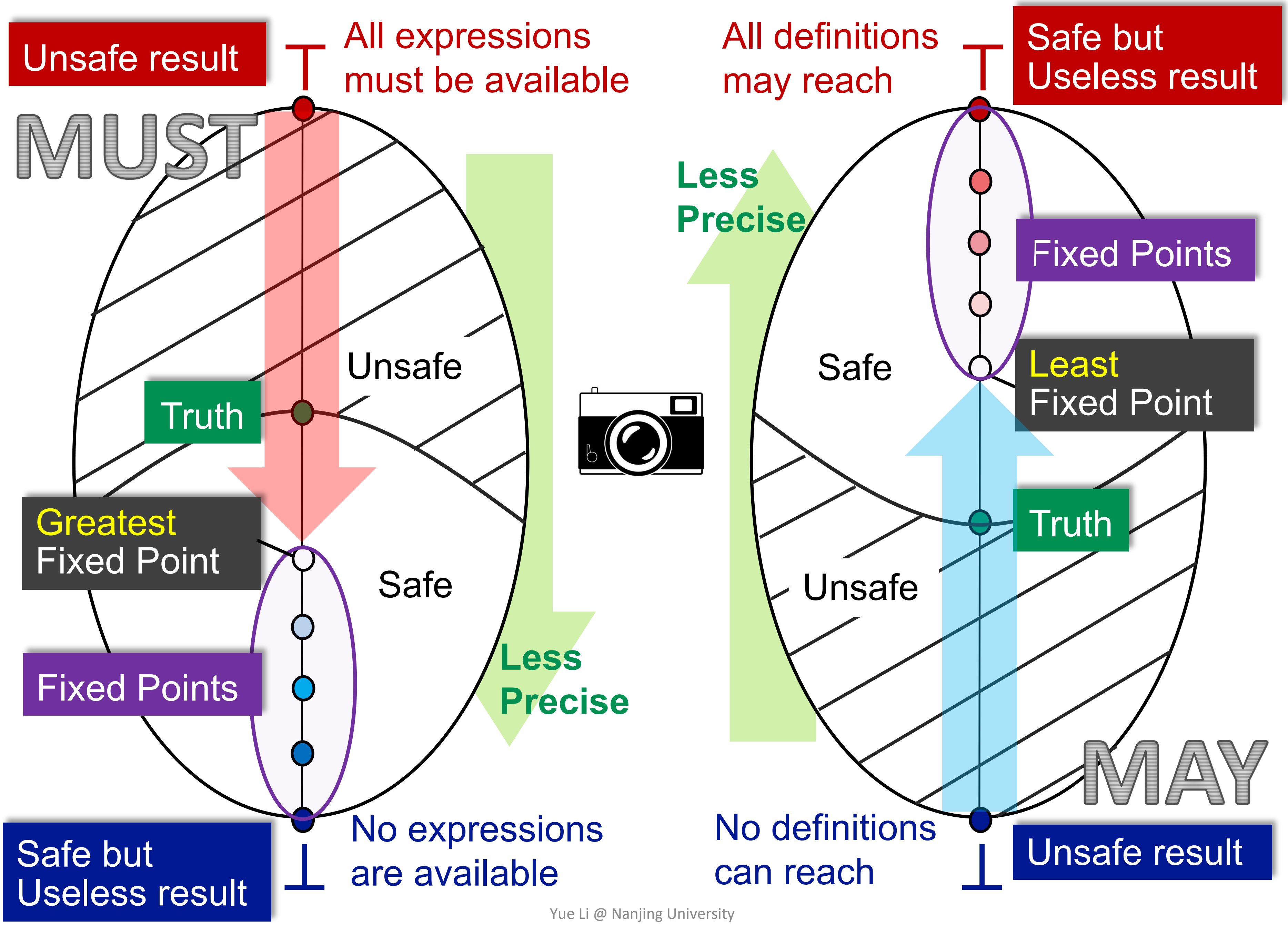


Another view to explain greatest/least fixed point?
("minimal step" by meet/join)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How Precise Is Our Solution?

- Meet-Over-All-Paths Solution (MOP)

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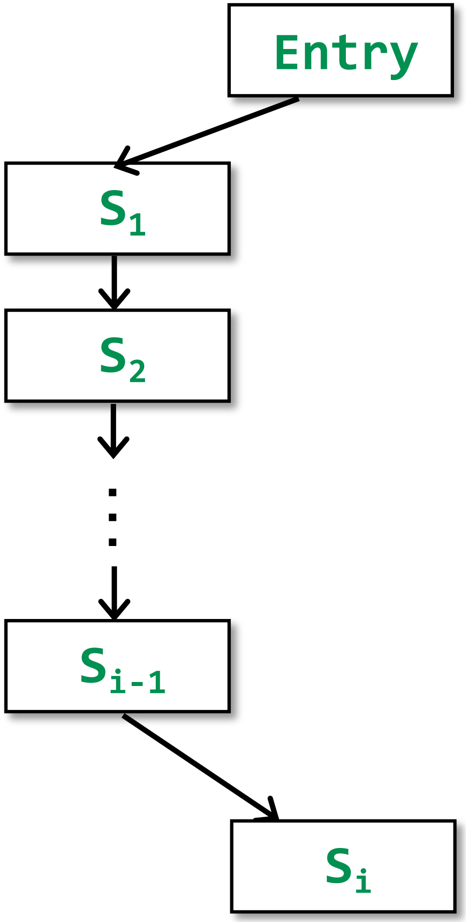
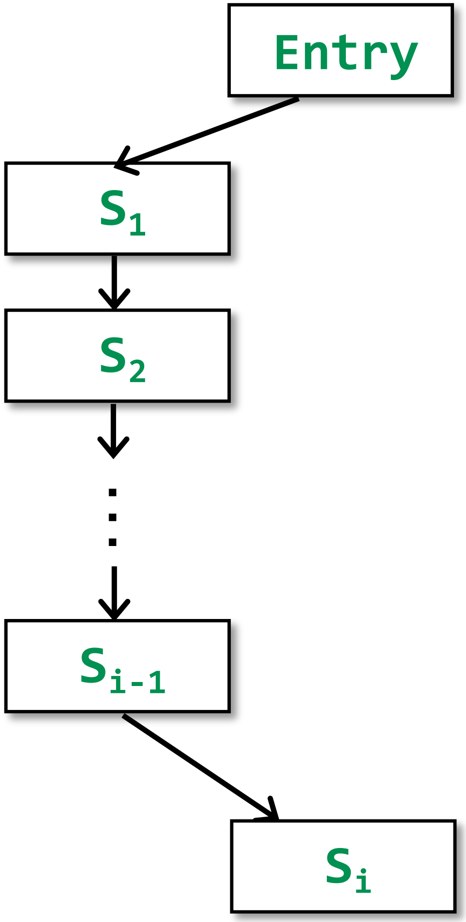


P = Entry → S_1 → S_2 → … → S_i

How Precise Is Our Solution?

- Meet-Over-All-Paths Solution (MOP)



P = Entry → S_1 → S_2 → … → S_i

Transfer function F_P for a path P (from Entry to S_i) is a composition of transfer functions for all statements on that path: f_{S1}, f_{S2}, …, f_{Si-1}

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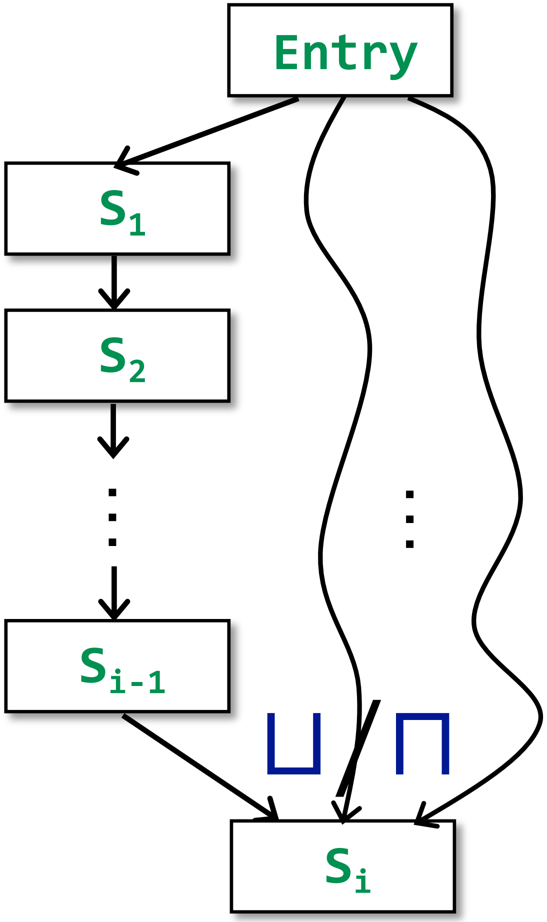
- Meet-Over-All-Paths Solution (MOP)



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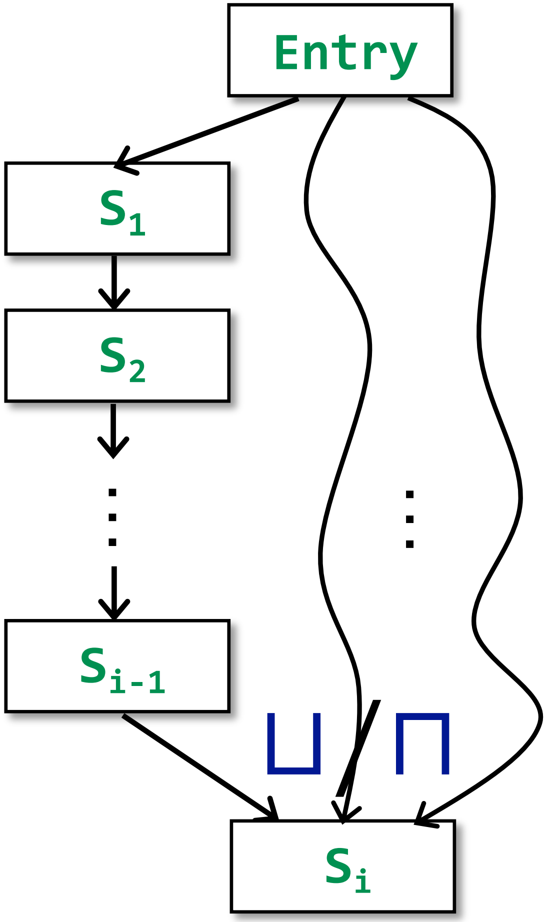
$$\boxed{\text{MOP}[S_i] = \underset{\textit{A path } P \textit{ from Entry to } S_i}{\bigsqcup / \bigsqcap} F_P(\text{OUT}[\text{Entry}])}$$

How Precise Is Our Solution?

- Meet-Over-All-Paths Solution (MOP)



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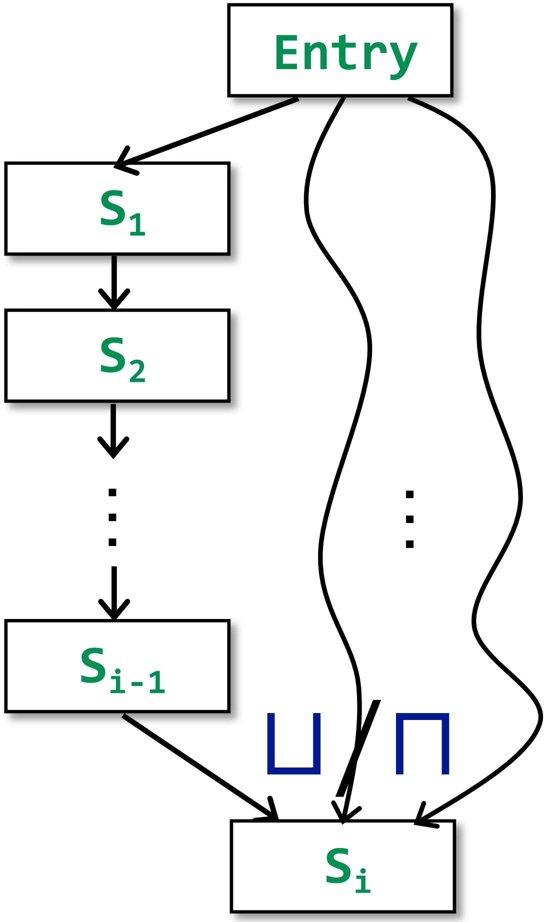
MOP computes the data-flow values at the end of each path and apply join / meet operator to these values to find their lub / glb

How Precise Is Our Solution?

- Meet-Over-All-Paths Solution (MOP)



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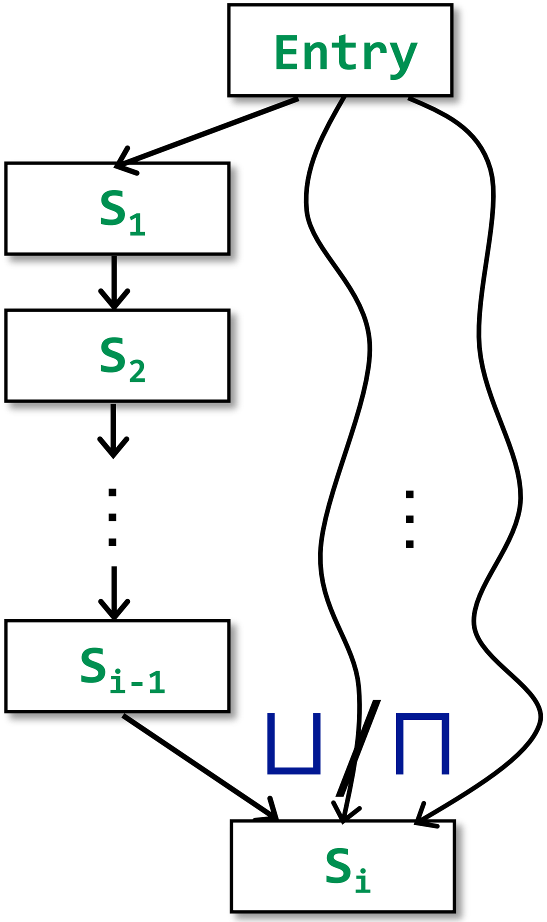

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Unbounded, and not enumerable → **impractical**

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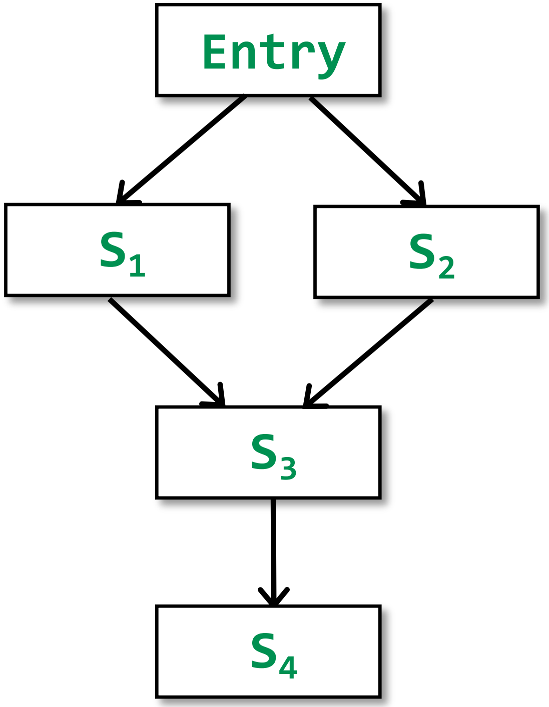

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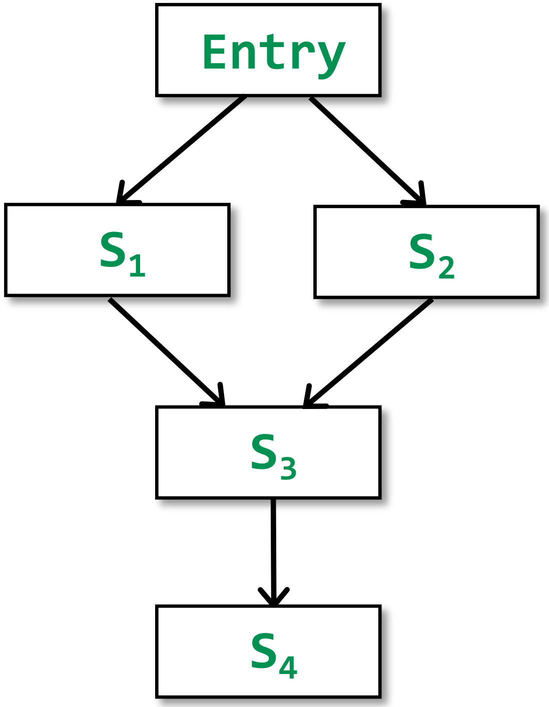

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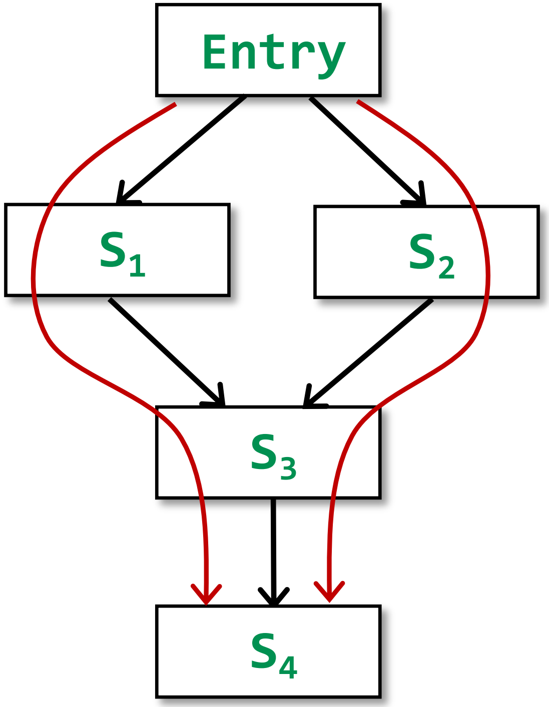

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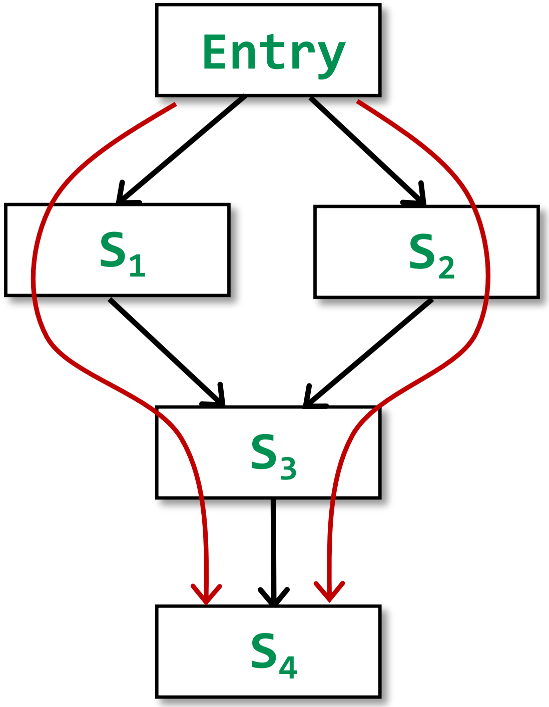

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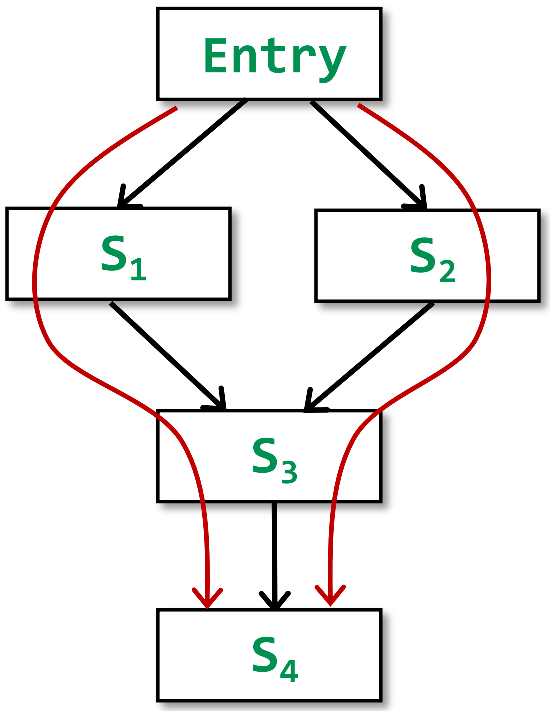

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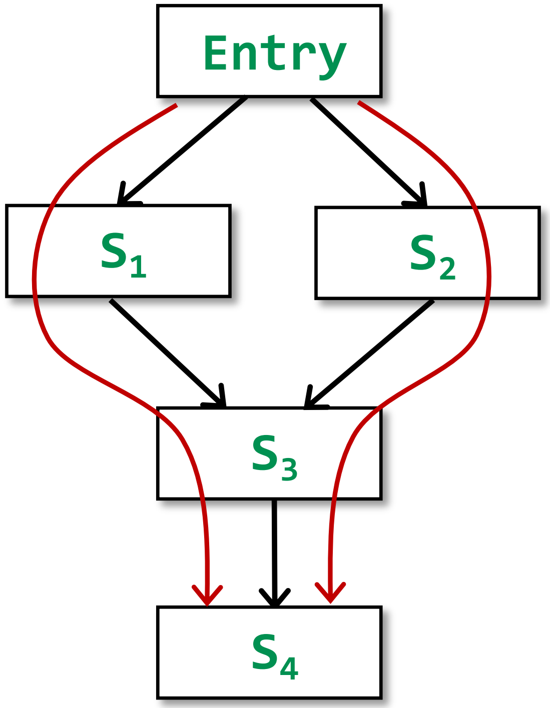

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But some analyses are not distributive

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Constant Propagation – Lattice

- Domain of the values V

- Meet Operator ⊓

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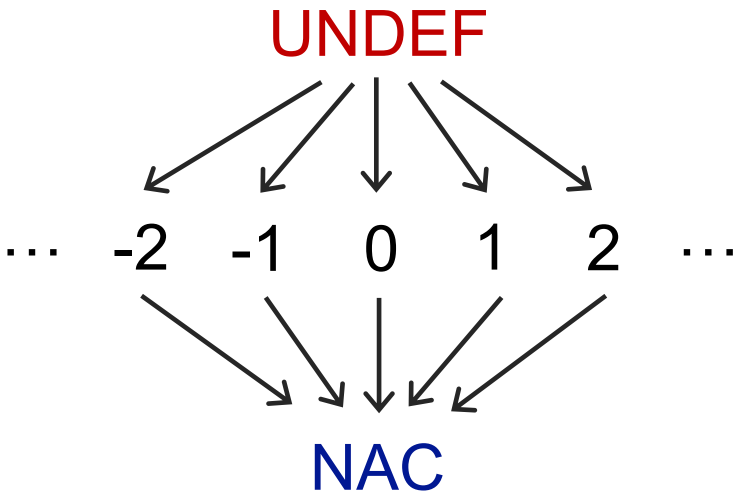

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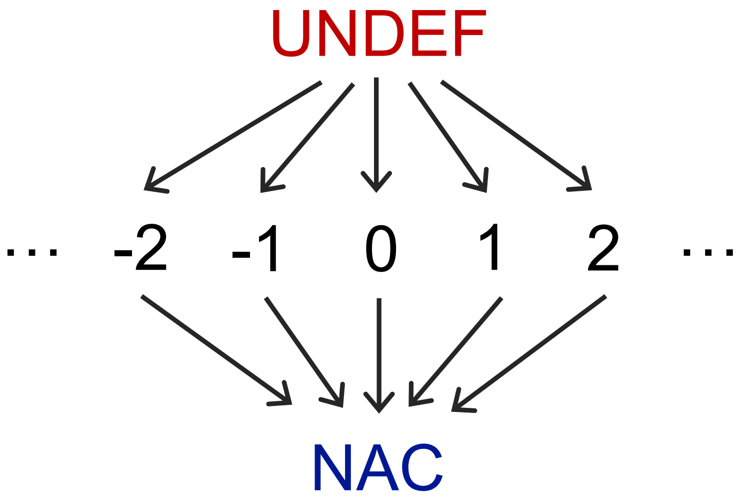

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NAC ⊓ v = NAC

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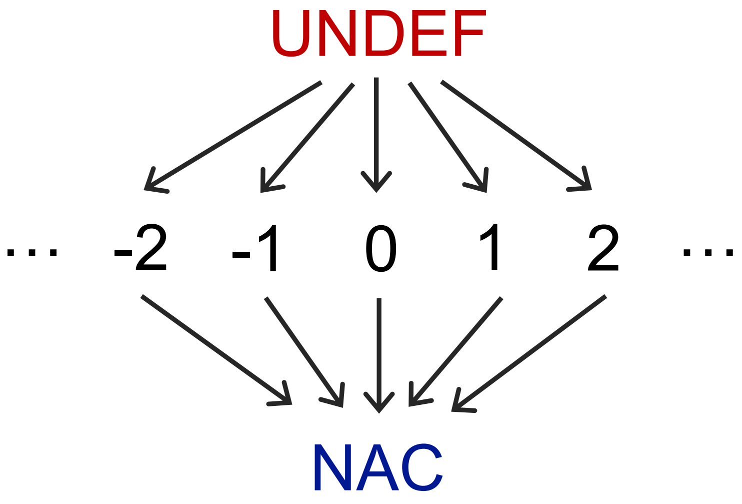

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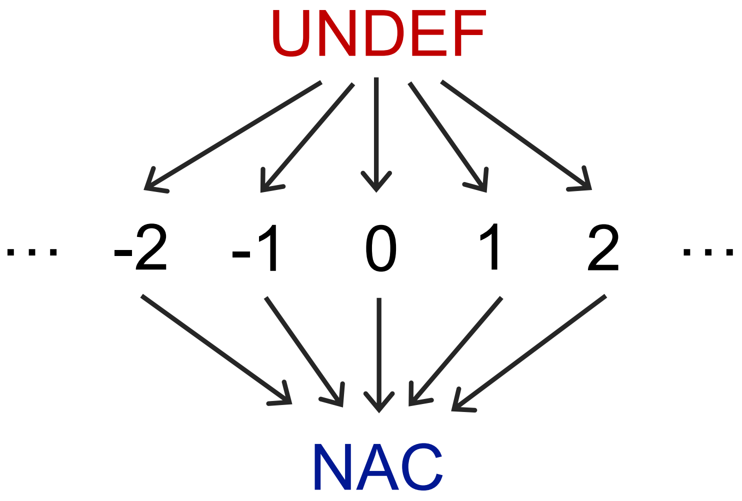

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Uninitialized variables are not the focus in our constant propagation analysis

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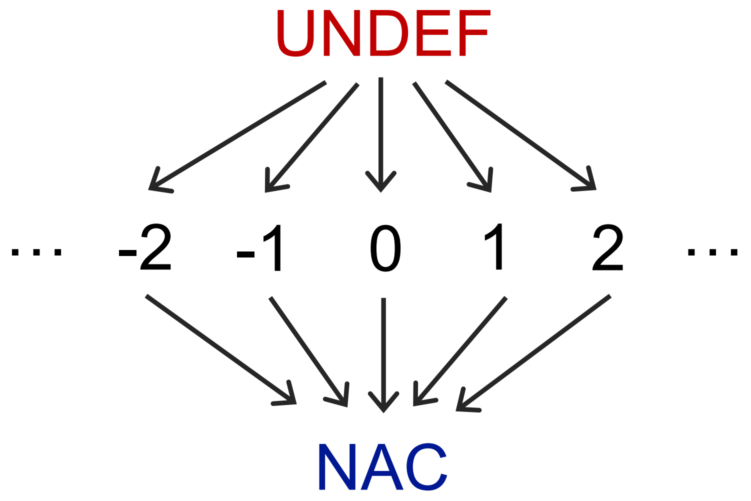

UNDEF

… -2 -1 0 1 2 …

NAC

- Meet Operator $\sqcap$

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$c \sqcap v$ = ?

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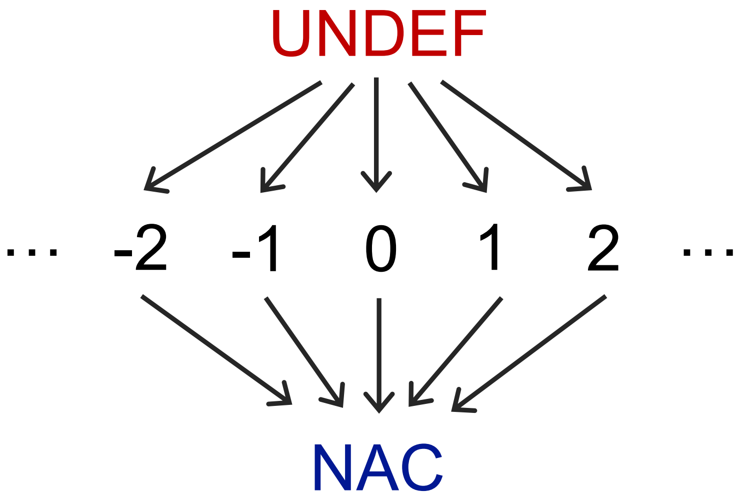

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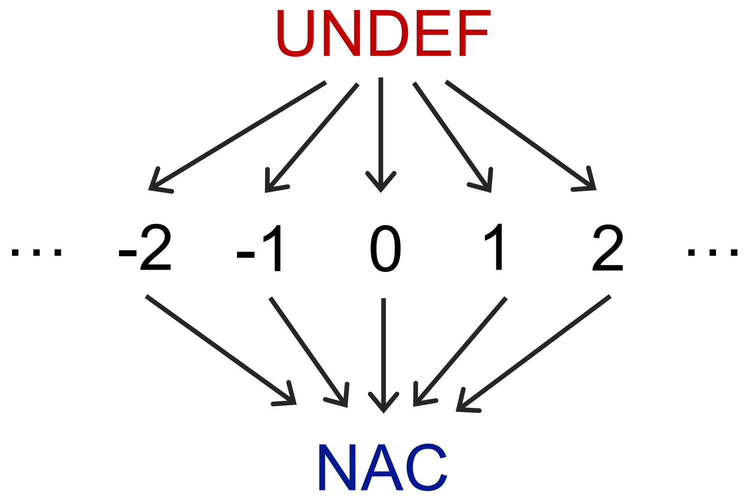

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$c \sqcap v = ?$

- $c \sqcap c = c$
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At each path confluence PC, we should apply “meet” for all variables in the incoming data-flow values at that PC

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Given a statement s: x = …, we define its transfer function F as

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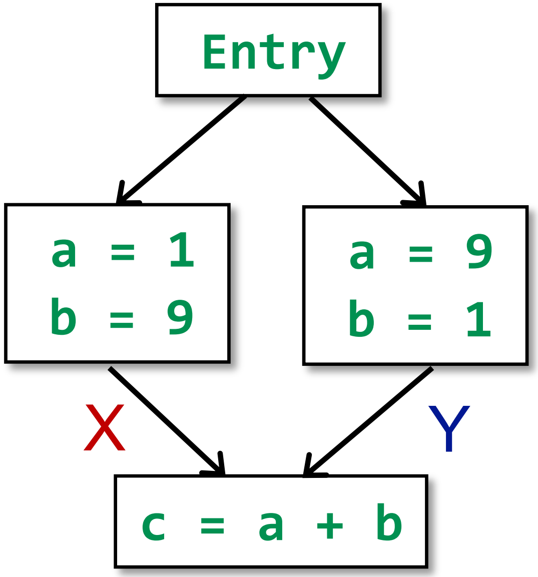
(if s is not an assignment statement, F is the identity function)

Constant Propagation – Nondistributivity



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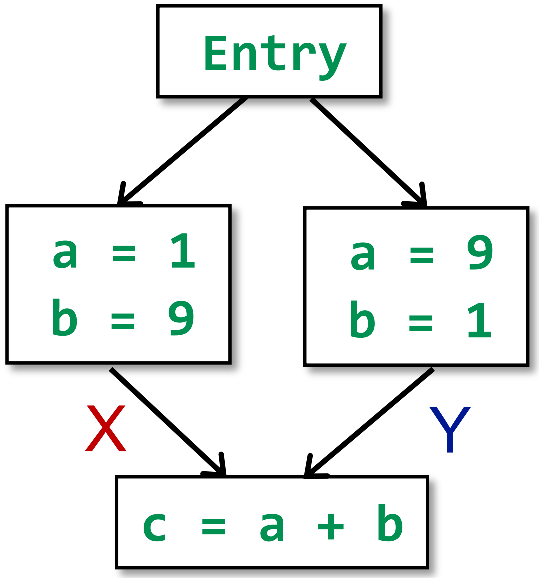
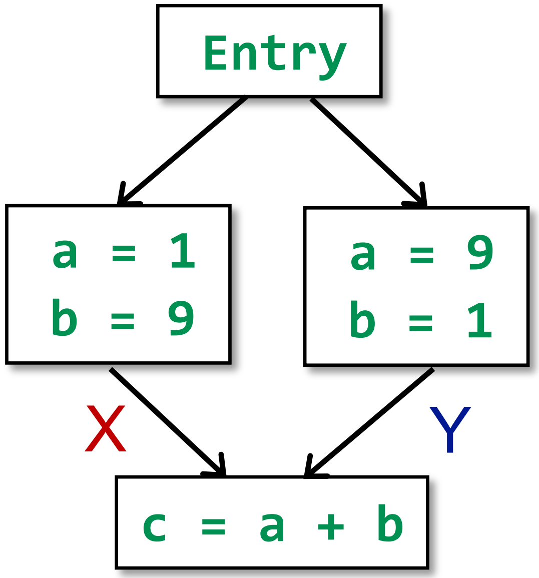


$F(X \sqcap Y) =$

$F(X) \sqcap F(Y) =$

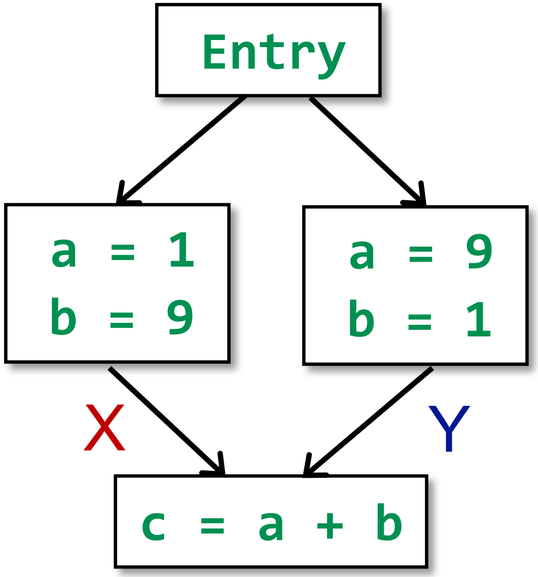
Constant Propagation – Nondistributivity



$F(X \sqcap Y) = \{(a, NAC), (b, NAC), (c, NAC)\}$

$F(X) \sqcap F(Y) =$

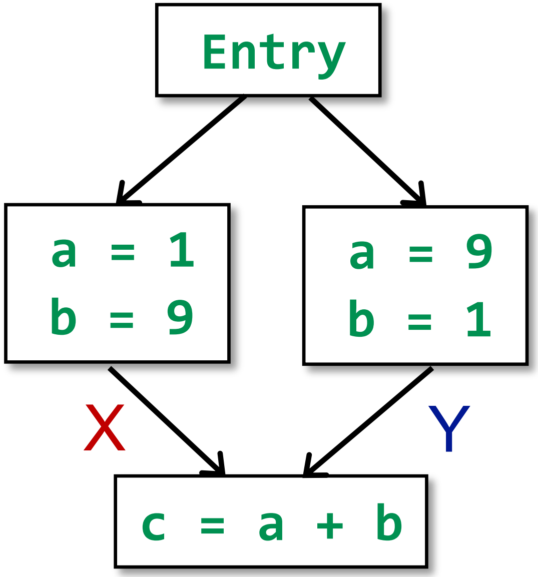
Constant Propagation – Nondistributivity



$F(X \sqcap Y) = \{(a, NAC), (b, NAC), (c, NAC)\}$

$F(X) \sqcap F(Y) = \{(a, NAC), (b, NAC), (c, 10)\}$

Constant Propagation – Nondistributivity

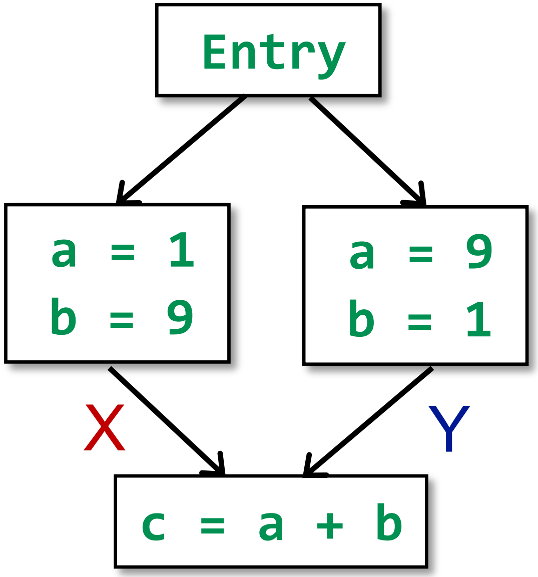


$$F(X \sqcap Y) = \{(a, NAC), (b, NAC), (c, NAC)\}$$

$$F(X) \sqcap F(Y) = \{(a, NAC), (b, NAC), (c, 10)\}$$

$$F(X \sqcap Y) \neq F(X) \sqcap F(Y)$$

Constant Propagation – Nondistributivity



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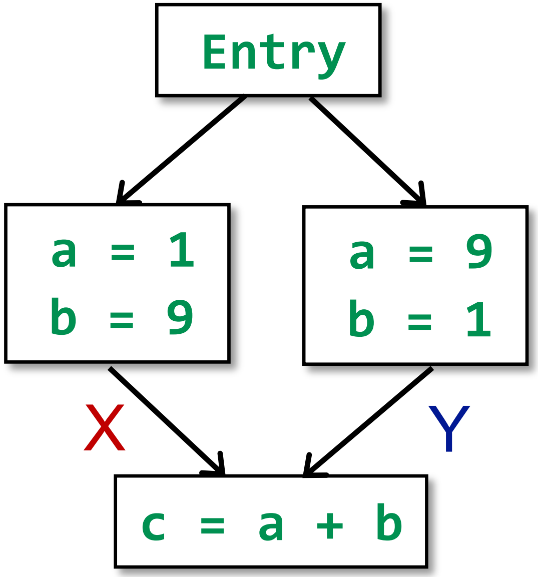
$$F(X \sqcap Y) \neq F(X) \sqcap F(Y)$$

$$F(X \sqcap Y) \sqsubseteq F(X) \sqcap F(Y)$$

Constant Propagation – Nondistributivity



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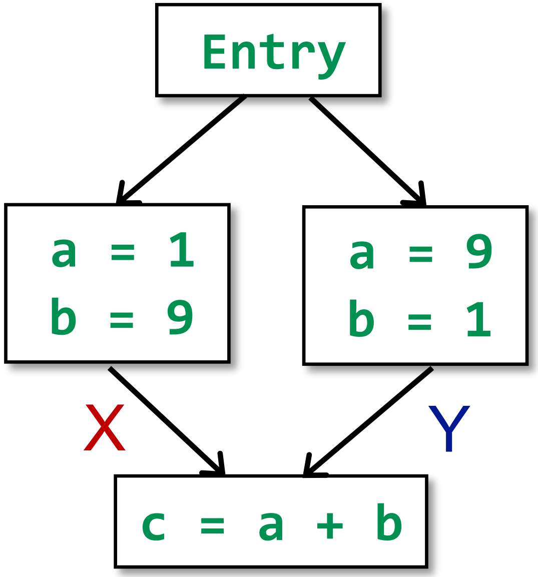
Show our constant propagation analysis is monotonic

Constant Propagation – Nondistributivity



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Show our constant propagation analysis is monotonic

Worklist Algorithm,

an optimization of Iterative Algorithm

Review Iterative Algorithm for May & Forward Analysis

INPUT: CFG ($kill_B$ and gen_B computed for each basic block *B*)

OUTPUT: IN[*B*] and OUT[*B*] for each basic block *B*

METHOD:

```
OUT[entry] = ∅;
for (each basic block B\entry)
    OUT[B] = ∅;
while (changes to any OUT occur)
    for (each basic block B\entry) {
        IN[B] = $\sqcup_{P \text{ a predecessor of } B}$ OUT[P];
        OUT[B] = $gen_B$ U (IN[B] - $kill_B$);
    }
```

Worklist Algorithm

Forward Analysis

```
$OUT[entry] = \emptyset$;
for (each basic block $B \setminus entry$)
    $OUT[B] = \emptyset$;
Worklist ← all basic blocks
while (Worklist is not empty)
    Pick a basic block $B$ from Worklist
    old_OUT = $OUT[B]$
    $IN[B] = \bigsqcup_{P \text{ a predecessor of } B} OUT[P]$;
    $OUT[B] = gen_B \cup (IN[B] - kill_B)$;
    if (old_OUT $\neq$ $OUT[B]$)
        Add all successors of $B$ to Worklist
```

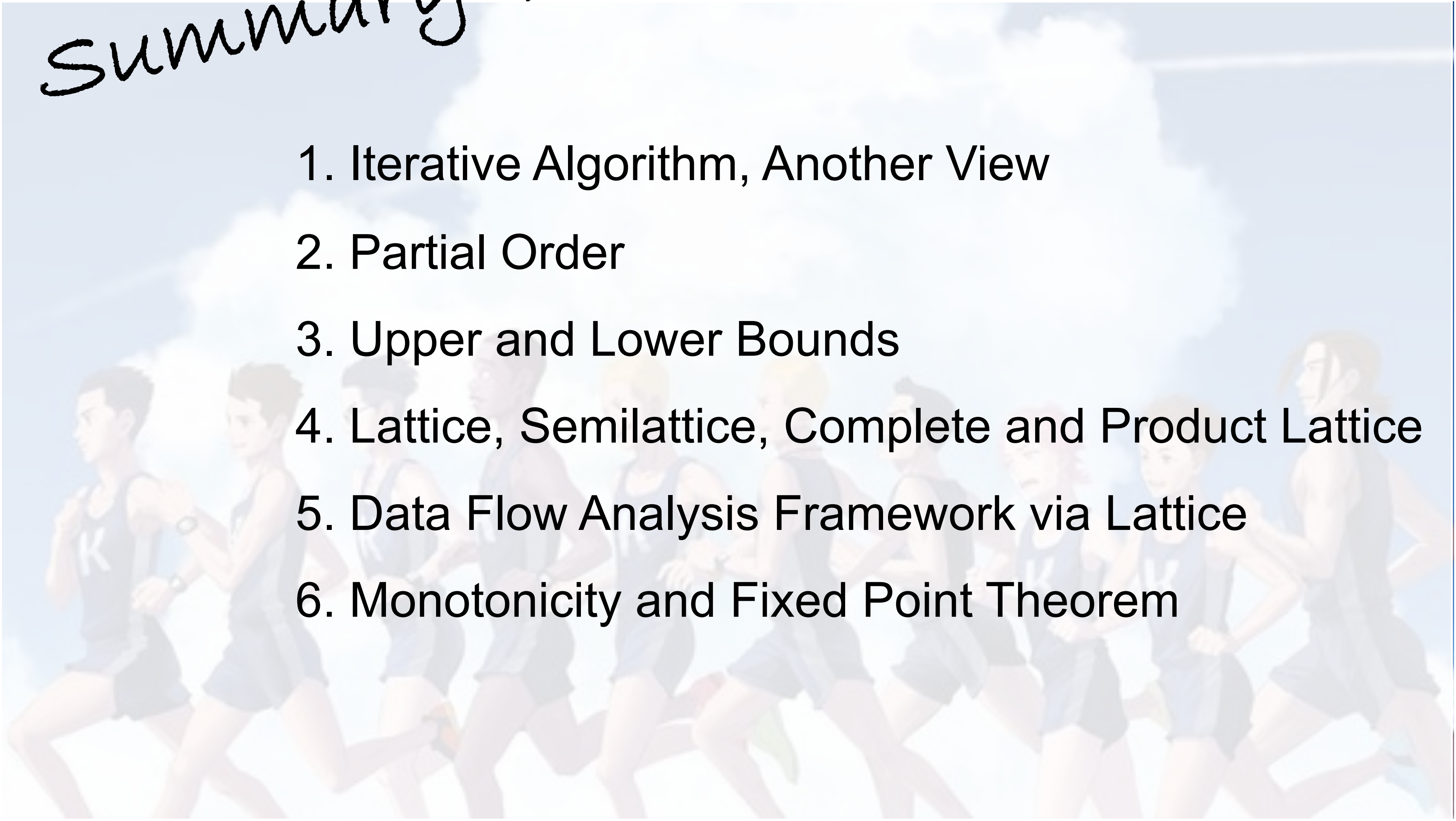
Worklist Algorithm

Forward Analysis

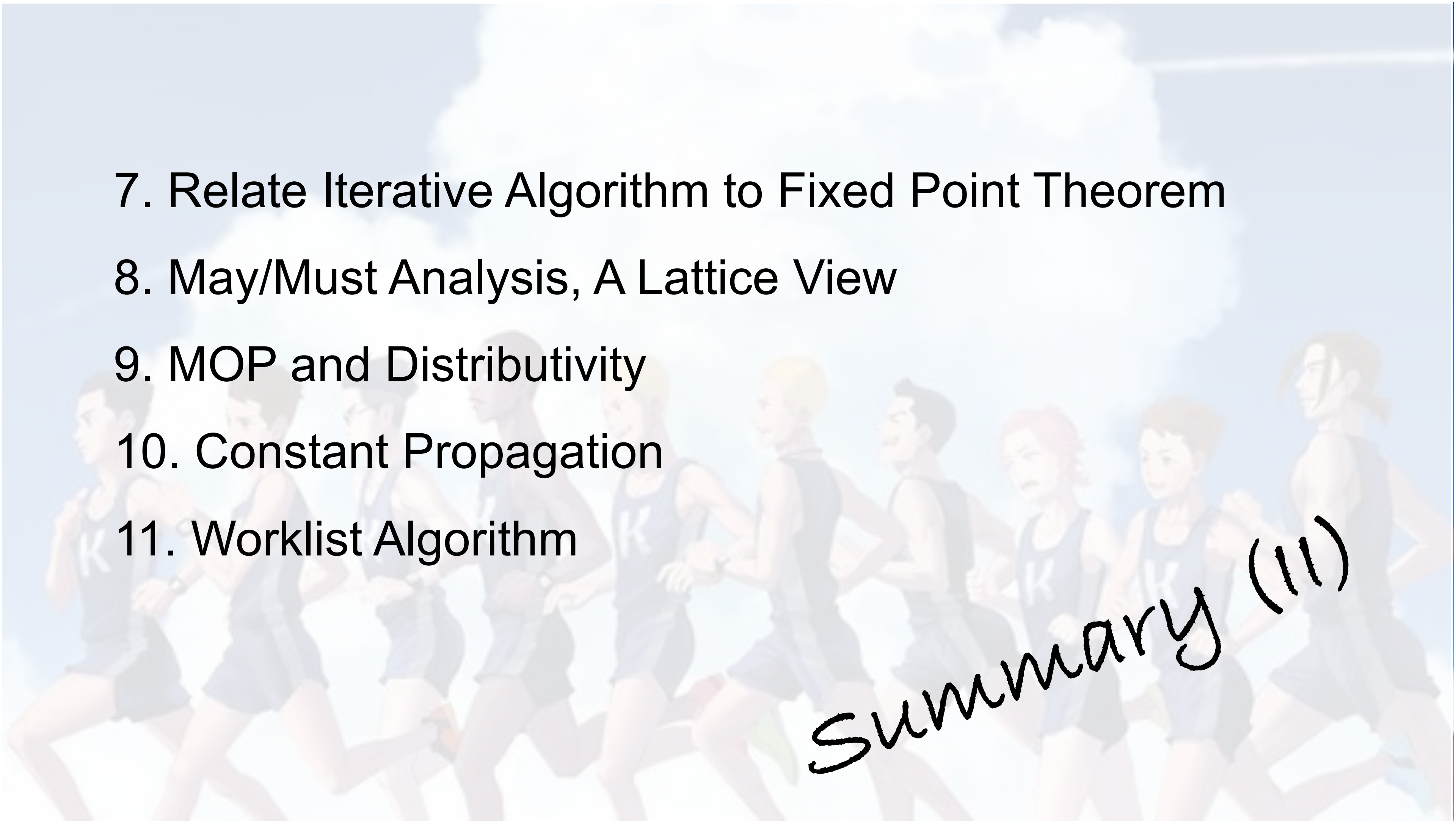
```
OUT[entry] = ∅;
for (each basic block B\entry)
    OUT[B] = ∅;
Worklist ← all basic blocks
while (Worklist is not empty)
    Pick a basic block B from Worklist
    old_OUT = OUT[B]
    IN[B] = $\bigsqcup_{P \text{ a predecessor of } B}$ OUT[P];
    OUT[B] = $gen_B \cup (IN[B] - kill_B)$;
    if (old_OUT ≠ OUT[B])
        Add all successors of B to Worklist
```

OUT will not change if IN does not change

Summary (I)



1. Iterative Algorithm, Another View
2. Partial Order
3. Upper and Lower Bounds
4. Lattice, Semilattice, Complete and Product Lattice
5. Data Flow Analysis Framework via Lattice
6. Monotonicity and Fixed Point Theorem



7. Relate Iterative Algorithm to Fixed Point Theorem
8. May/Must Analysis, A Lattice View
9. MOP and Distributivity
10. Constant Propagation
11. Worklist Algorithm

Summary (II)

The X You Need To Understand in This Lecture

- Understand the functional view of iterative algorithm
- The definitions of lattice and complete lattice
- Understand the fixed-point theorem
- How to summarize may and must analyses in lattices
- The relation between MOP and the solution produced by the iterative algorithm
- Constant propagation analysis
- Worklist algorithm

Assignment Two:
Constant propagation and worklist solver